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ABSTRACT

This handbook was designed to be used in staff development programs for teachers of fourth and fifth grade mathematics. The program strives to make the teaching and learning of mathematics more meaningful by emphasizing process experiences. Good questioning techniques, problem exploration strategies and the use of manipulative materials are stressed in this program. This project addresses instruction in grades 4 and 5 in an integrated fashion. The design was chosen in an attempt to enrich participants' understanding of the way topics develop through these grades, while at the same time providing the teachers with an opportunity to share insights about bridging students' mathematics learning experiences in the two grades. Included in the guide are background information for the teacher, teaching units and activities, and reproducible handouts and worksheets. (PK)

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MATHEMATICS INSTRUCTION Grades 4 & 5

Staff Development 1987-1988

New York City Board of Education

Division of Curriculum and Instruction

Office of Curriculum Development and Support

Mathematics Unit

SE 048 863

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FOREWORD

We are very enthusiastic about this staff development initiative, **MATHEMATICS INSTRUCTION IN GRADES 4 AND 5,** because it is a continuation of our efforts to make the teaching and learning of mathematics more meaningful by emphasizing process experiences. Good questioning techniques, problem exploration strategies and the use of manipulative materials are stressed in this program. Research says that through attention to meaning, mathematics is seen as a close knit system of understandable ideas, principles and processes. We find that mathematics instruction is most effective when both teachers and students see sense in what they teach and learn.

This project is unique in that it addresses instruction in Grades 4 and 5 in an integrated fashion. This design was chosen to enrich participants' understanding of the way topics develop through these grades, while at the same time providing the teachers with an opportunity to share insights about bridging students' mathematics learning experiences in the two grades.

We hope that the implementation of this staff development initiative will not only increase the students' basic skills competency, but will also serve the ultimate goal of developing students' mathematics power to the fullest.

Charlotte Frank

CHARLOTTE FRANK
Executive Director

ACKNOWLEDGMENTS

The **Mathematics Instruction in Grades 4 and 5** Staff Development Program for 1987-88 was developed as a project of the Mathematics Unit of the Division of Curriculum and Instruction, Charlotte Frank, Executive Director.

Morris Freedman, Deputy Executive Director of Office of Curriculum Development and Support, provided overall supervision of this project.

This project was developed under the direction of Susan Zakaluk, Director of Mathematics.

The program was cooperatively planned by the Mathematics Unit, the C.I.M.S. Mathematics Group and the District Mathematics Coordinators; special contributions were made by:

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Additional materials were contributed by Ira Ewen, Director of the Reasoning Skills Project.

The philosophy of selecting and designing staff development activities on the basis of contemporary research findings, which guides this project, is a continuation of the concept originally expressed in Gates to Learning: Mathematics Leadership Seminar, which was prepared by Marvin Schneider and Maria DeSalvio under the supervision of Irwin Kaufman, and extended in "Research Says...", which was prepared by Lorraine Largmann. Excerpts from the latter document are integrated in the text of this manual.

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MATHEMATICS UNIT: POLICY ON PROBLEM SOLVING

A major goal of education is to prepare the student to succeed in unfamiliar situations; this ability is far more valuable (and far less easy to attain) than the ability to repeat the application of a procedure in which one has been tutored and rehearsed to a situation which is completely familiar. In the first case, the student uses courage, determination, creativity and ingenuity to combine and extend concepts and techniques previously learned in a new way; in the second case, the student depends on memory and attention to detail to duplicate a pattern internalized previously: there is no new growth. In the first case, the student actively encounters a problem, and may solve it; in the second case, although he may display a solution, the student has merely performed an exercise.

Research reveals two effective instructional techniques which help students grow in their ability to create and discover new ways to combine and apply concepts they have learned:

(1) The teacher should help the student develop a language for describing and thinking about problem solving, especially those strategies (e.g., working backward, considering a simpler analogous problem) which are effective in a wide variety of situations.

(2) The teacher should help the student develop the habit of reflection: students should be led to think about new ideas they encounter; summarizing, comparing and extending concepts they are taught, students will better internalize them. The use of a student learning journal as a regular component of the instructional process is highly effective for this goal, and is strongly recommended.

Students should be encouraged to create questions and invent problems. They should wonder about mathematics, and raise questions like: Why does the denominator change when we multiply fractions, but not when we add them? When should I use the median rather than the mean to describe "average" performance?

Beginning in 1987, each newly revised Scope and Sequence of Instructional Objectives included "Problem Solving Strategies" listed at the end of each unit. These are techniques which are useful in a wide variety of situations, both in mathematics and in other subjects as well. Here is a partial list of these important strategies:

STRATEGIES FOR PROBLEM EXPLORATION

1. **STATE THE GOALS OF A PROBLEM CLEARLY**
The problem solver should internalize the problem so that he is able to state the goals of the problem clearly without looking at notes. He should also be able to explain what pertinent information is available, and weed out irrelevant data.

2. **MAINTAIN AN OVERVIEW**
The problem solver should maintain an awareness of the goals and context of the problem, and be able to explain how what he is doing at a given time bears on the solution.

3. **WORK BACKWARD**
In working backward, the problem solver starts with the goal, and determines the condition or conditions that would be necessary to achieve it. There may be several different paths leading backward from the goal. The student must learn to identify subsidiary goals.

4. **RESTATE THE PROBLEM TO ELIMINATE AMBIGUITIES AND CLARIFY OBJECTIVES**
Human communication tends to be imprecise. By expressing the problem in his own words, the student may discover terms which are subject to more than one interpretation or which require definition. He may discover a new perspective on the problem that facilitates its solution.

5. **LOOK FOR HIDDEN ASSUMPTIONS**
The student should learn to identify and explain unstated conditions which are part of the problem.

6. **ADOPT A DIFFERENT POINT OF VIEW**
The problem solver should look for and explore all reasonable interpretations of problems being studied. One of the interpretations may suggest an analogous problem which has been studied before, or which is easier to solve.

7. **CONSIDER A RELATED BUT SIMPLER PROBLEM**
To gain insight into a complex problem, the problem solver may create a new problem which resembles it but is easier to solve (for example, replacing decimal numerals with whole number approximations, or substituting small numbers for large ones). The solution process for the simpler problem will serve as a model for solving the original problem.

8. **CONSIDER SPECIAL CASES**
The problem solver may gain insight into a situation by temporarily introducing special characteristics (for example, by seeing what would happen if a given rectangle were actually a square); later, he removes the conditions he inserted and studies how their removal affects the conclusions which hold in the special case.

9. **CONSIDER EXTREME CASES**
The problem solver should test a conjecture by looking at both extreme ends of its range of expected applicability, for example, by seeing whether it holds for both very large and very small values; for circles and for triangles; for parallel and for perpendicular lines.

10. USE ANALOGY
The problem solver should look for problems similar in some way to the one he is facing, and attempt to apply methods and insights about one to the other. An analogy is an inference that if two or more things agree in some respects, they will probably agree in others.

11. FIND A REAL OR AN ABSTRACT MODEL FOR A PROBLEM
The problem solver identifies or creates a simple image which preserves the structure of the situation he is studying. He may gain better understanding of a problem about the volume of a cube by studying a cardboard box, or associate a problem about forming a committee with selecting letters from a box to form a word.

12. CARRY OUT A SIMULATION
The problem solver may improve his understanding of a problem by acting it out. Using a model, the problem solver may obtain experimental results that give insight into theoretical relationships.

13. COMPARE AND CONTRAST
The problem solver relates new concepts, objects and situations to familiar ones, and identifies similarities and differences. Comparison and contrast together expose the unique features and properties of the new item.

14. SYSTEMATICALLY ACCOUNT FOR ALL POSSIBILITIES
This strategy is useful when a complex problem can be broken down into separate cases to be considered one by one. Also, if the number of possible solutions to a problem is limited, the correct solution or solutions can sometimes be found by eliminating the others one by one.

15. CONSIDER THE CONSEQUENCES OF AN IDEA
Often it becomes obvious that a proposed idea cannot be true because it leads to a consequence which is known to be false. The problem solver should always explore the consequences of an idea he has proved; a result beyond the original goal may be a more important and powerful idea than the one originally sought.

16. ELIMINATE THE IMPOSSIBLE BUT NOT THE IMPROBABLE
The problem solver should naturally seek a counterexample of a new conjecture, but should eliminate only those cases which he can demonstrate are impossible. Situations which are rare are often enormously interesting.

17. LOOK FOR A PATTERN
Problem solvers should learn to look for an element or relationship that repeats or extends itself in a predictable way. When a pattern is discovered the student should attempt to find a reason why it exists and consider whether it must continue.

18. USE INFERENCE
The problem solver should use logical reasoning to draw conclusions from evidence.

19. GENERALIZE CONCLUSIONS CAREFULLY
Problem solvers should form conjectures which are consistent with data, but should be aware when they are dealing with supposition rather than fact. They should try to extend patterns and expand ideas, and see if they can prove their generalizations.

20. RECOGNIZE AND USE SYMMETRY
Symmetry in a structure often helps us to simplify a procedure or a proof. The knowledge that, because of symmetry, a statement will have the same truth value if certain elements are exchanged can guide the problem solver in the construction of reasonable conjectures, and the elimination of impossible ones.

21. MAKE A DIAGRAM
Diagrams clarify and consolidate information and may help the problem solver discover new connections between the components of a problem. They provide a concise summary of relationships.

22. KEEP GOOD RECORDS
Data should be organized to allow easy access and overview. Records of attempts at solution, whether successful or not, should be reviewed; insight may be gained in either case. Charts and graphs of various types should be used, as each may reveal a different aspect of the problem more clearly.

23. ESTIMATE
Finding an approximate solution to a problem creates a framework of expectation in which later work can be judged. Comparison with the estimated value may reveal a flaw in the solution process or the occurrence of an error in calculation.

24. USE TRIAL-AND-ERROR
The problem solver makes a conjecture, investigates consequences, compares the results to the goal, and then modifies the conjecture to approach the goal more nearly. By repeating this process, he may discover the solution.

25. PRIORITIZE GOALS
Sometimes the goals of a problem conflict. (The line cannot bisect the angle if it must pass through the given point.) The problem solver must learn to distinguish between goals which are essential and those which are merely desired.

26. SEQUENCE IDEAS
The problem solver must be able to identify the relationship between ideas; especially, he must see which ideas follow from others and which are independent.

27. FIND AN ALTERNATIVE SOLUTION
A different solution may have advantages over those which are known. One may be more efficient, another can be generalized, a third is easiest to remember. Realizing that two different processes can be used to solve a problem leads the problem solver to look for a relationship between the processes.

We have associated a specific set of strategies with each unit in the course, because each unit provides the teacher with the opportunity to model and identify some strategies more easily than others.

Every grade's Scope and Sequence contains a specific set of Problem Solving units, with objectives which should be taught to the student in that grade. This is one of the content strands of the mathematics curriculum which stretches through all grades. The objectives included in Strand 13 in each grade should be mastered by the student so that they become a part of the students growing set of resources to demystify unfamiliar situations.

The CIMS Teacher's Manual for this grade contains specific suggestions about activities and discussions that will raise students' awareness of these strategies. It also suggests specific points in instruction where you can create the opportunity for students to compare problems of their own.

Further information about the use of student learning journals to support mathematics instruction is available through the Division of Curriculum and Instruction.

We hope that you will use and enjoy these materials, and will share both your problems and successes with us.

The Mathematics Unit
The Reasoning Unit

"A mathematician who is not also something of a poet
will never be a complete mathematician."

- Karl Weierstrass

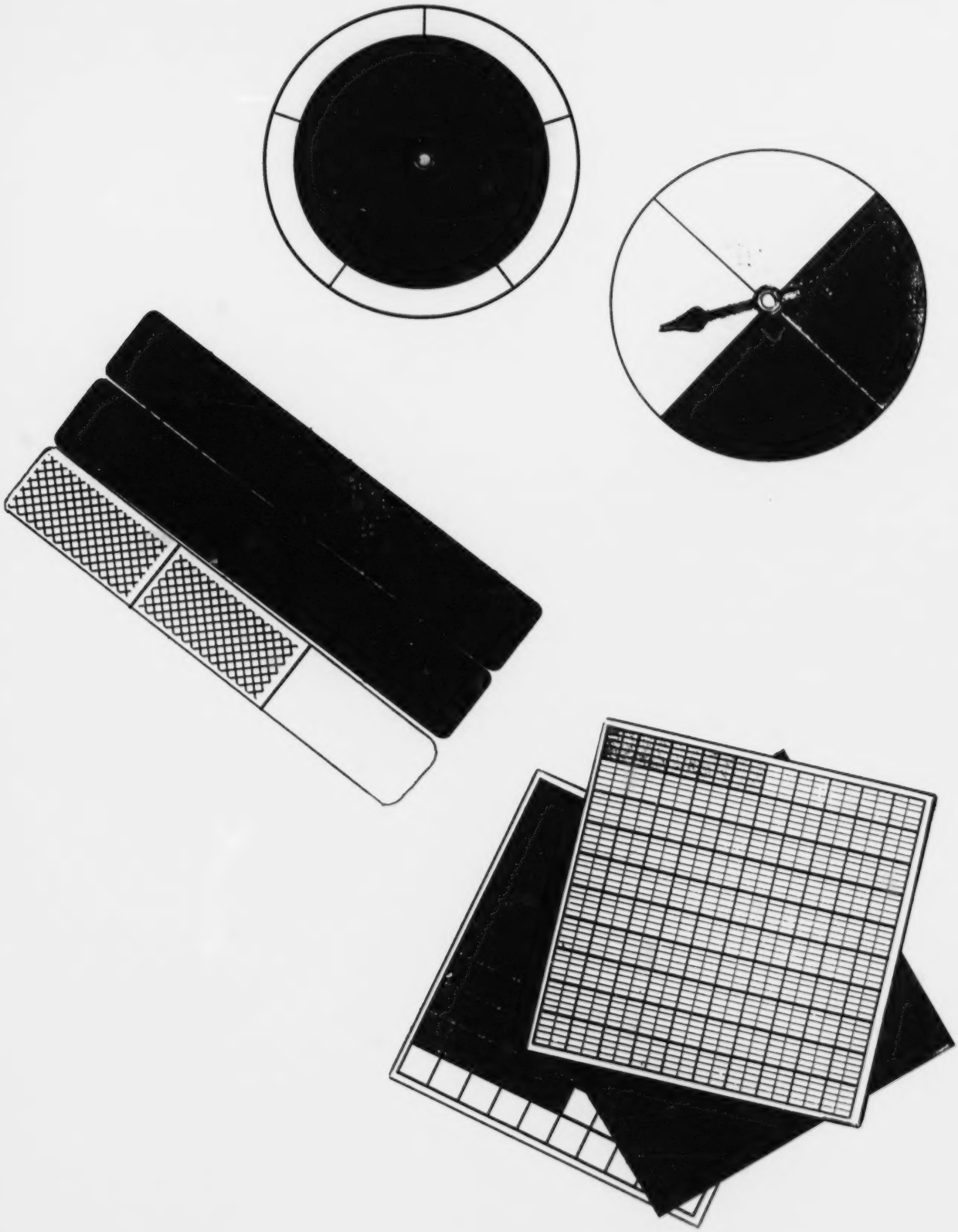
"We often find that a good question
is more important than a good answer."

- Richard Bellman

MATHEMATICS STAFF DEVELOPMENT

GRADES 4 AND 5

PROGRAM



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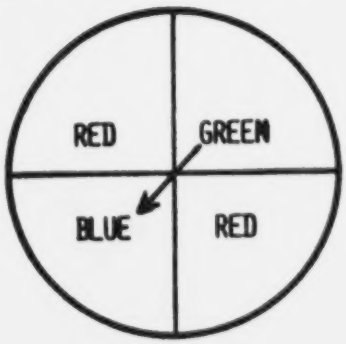
MATHEMATICS STAFF DEVELOPMENT

GRADES 4 AND 5
PROGRAM

DAY 1:

PROBLEM OF THE DAY



If you were to spin this spinner, what would be the probability that the arrow would land on:

Blue?
Red?
Green?
Yellow?

I. **NEW YORK STATE'S RECOMMENDATIONS AND NEW YORK CITY'S RESPONSE**

- *The New York State Syllabus
- *The New York City Scope and Sequence
- *The New York City Mathematics Test
- *N.Y.C. Mathematics Policy: Focus on Problem Solving
- *Professional Mathematics Support Organizations

PARTICIPANTS WILL BE ABLE TO:

- *Identify and distinguish the New York State Syllabus,New York City Scope and Sequence, N.Y.C. Mathematics Test, N.Y.C. Policy for Mathematics
- *Identify resources to support their instructional program

DAY 1 - continued . . .

II. **RATIONALE FOR THE USE OF MANIPULATIVE MATERIALS FOR CLASSROOM INSTRUCTION**

- *Manipulative activities that support thinking and reasoning skills
- *Overview of manipulative materials kit

PARTICIPANTS WILL BE ABLE TO:

- *Provide a research based rationale for the student's use of manipulatives in learning mathematics
- *Explain how manipulatives are used to increase student's thinking/reasoning skills
- *Provide examples of the use of manipulatives to increase student's concept, mastery, and reasoning skills

III. **ORGANIZING FOR EFFECTIVE INSTRUCTION**

- *Classroom management techniques to support productive student interaction and facilitate the use of manipulative materials
- *Introduction of "A Collection of Math Lessons" - by Marilyn Burns

PARTICIPANTS WILL BE ABLE TO :

- *Identify Classroom management techniques for the use of manipulative materials

IV. **DEVELOPING INSTRUCTIONAL APPROACHES FOR TEACHING PROBABILITY**

- *Probability strand of the N.Y.C. Curriculum
- *Fundamental Probability concepts
- *Classroom activities for teaching Probability: student experimentation

PARTICIPANTS WILL BE ABLE TO:

- * Plan lessons about Probability that incorporate the use of manipulative materials
- *Identify specific manipulatives which may be used in the teaching of Probability

V. **ADMINISTRATIVE MATTERS**

- *OP-85
- *Time Sheets
- *Attendance Forms

DAY 2:

WRITTEN ASSIGNMENT

Create a story to illustrate finding 1/2 of 2/3

I. STUDENT VERBALIZATION

*Language and Vocabulary
*Questioning techniques that promote student verbalization
*Learning logs

PARTICIPANTS WILL BE ABLE TO:

*Explain the need for using appropriate, correct mathematical language and vocabulary: oral and written
*Identify questioning techniques that promote student verbalization and understanding of mathematical concepts
*Explain the importance of student's creating their own problems and questions

II. LESSON PLANNING

*Components of a developmental mathematics lesson
*Planning for whole class, small group, and individual instruction
*Designing lessons to promote building problem solving approaches
*Excerpts from "A Collection of Math Lessons"

PARTICIPANTS WILL BE ABLE TO:

*Identify the components of a developmental math lesson
*Differentiate between whole class, small group, and individualized instruction

III. DEVELOPING INSTRUCTIONAL STRATEGIES FOR TEACHING FRACTIONS

*Fraction strand of the N.Y.C. Curriculum
*Fundamental fraction concepts
*Classroom activities for teaching fraction concepts

Day 2 - continued . . .

PARTICIPANTS WILL BE ABLE TO:

- *Plan lessons on fractions that incorporate the use of manipulative materials
- *Identify specific manipulatives which may be used in the teaching of fractions
- *Incorporate the use of the overhead projector in the teaching of manipulative materials for fraction

IV. SPIRALING HOMEWORK

- *What Works - Quantity vs. Quality of Homework
- *Rationale for Spiraling Homework
- *Process for Planning Spiraled Assignments
- * Homework planning on a weekly basis

PARTICIPANTS WILL BE ABLE TO:

- *Explain the benefits of spiraling homework
- *Explain the process of creating spiral homework assignments
- *Create spiral homework assignments

V. ADMINISTRATIVE MATTERS

- *Time Sheets
- *Attendance Forms

DAY 3:

WHAT QUESTIONS CAN YOU CREATE FROM SAM'S SPORTS SALE CIRCULAR?

I. ENCOURAGING STUDENTS TO ASK QUALITY QUESTIONS

- *What are quality questions?
- *Why should students learn to create questions?
- *How can we stimulate our students to pose good questions?

PARTICIPANTS WILL BE ABLE TO:

- *Explain the role of questioning in improving student thinking
- *Identify questioning techniques that promote student thinking

DAY 3 - continued . . .

II. **DEVELOPING INSTRUCTIONAL APPROACHES FOR TEACHING DECIMALS**

- *Decimal strand of the N.Y.C. Curriculum
- *Fundamental decimal concepts
- *Classroom activities for teaching decimal concepts

PARTICIPANTS WILL BE ABLE TO:

- * Plan lessons about decimals that incorporate the use of manipulative material
- *Identify specific manipulatives that may be used in the teaching of decimals
- *Incorporate the use of the overhead projector in the teaching of manipulative materials for decimals

III. **PROBLEM SOLVING**

- *Problem Solving throughout the Curriculum
- *Problem Solving strand of the N.Y.C. Curriculum
- *Classroom activities for teaching problem solving strategies

PARTICIPANTS WILL BE ABLE TO:

- *Describe how and why problem solving is incorporated throughout the math curriculum
- *Identify specific problem solving strategies
- *Develop instructional activities for teaching problem solving strategies

IV. **ADMINISTRATIVE MATTERS**

- *Time Sheets
- *Attendance Forms
- *Evaluations

FOLLOW-UP

- *Survey of participants (recommendations)

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THE NEW YORK CITY

MATHEMATICS TEST

GETTING READY FOR THE NEW YORK CITY MATHEMATICS TEST

TUESDAY, APRIL 7, 1987

Grades 2-4 Parts 1 and 2
Grades 5-8 Parts 1 - 4

WEDNESDAY, APRIL 8, 1987

Grades 2-4 Parts 3 and 4

In New York City public schools, students in the elementary and junior high schools are tested annually with standardized tests in reading and mathematics. The tests yield both norm-referenced and criterion-referenced scores that are reported to teachers, parents, and administrators. This testing program provides results useful to both administrative and instructional decision-makers in planning and implementing educational programs for students.

In mathematics, New York City administers a series of norm-referenced, objective-based mathematics tests to all students in grades 2 through 8. At the end of the customization process, a different level of the test will be administered at each grade. The test will be unique to New York City because it will have been customized to measure specific objectives found in the New York City mathematics curriculum. This test is customized by replacing shelf test items with new items that measure objectives in the New York City mathematics curriculum that were inadequately covered by the shelf test. About 60-75 percent of the items from the shelf test will be retained in the New York City test; this permits norm-referenced scores to be generated from the shelf test national norms. The balance of the items are newly created and will measure objectives not adequately covered by the original content of the shelf test. These items will appear in Part 4 (General Survey) on the Spring 1987 exam, but will not count toward student scores this year. Items from this group will be included in the regular test beginning in Spring 1988.

These brief guidelines, prepared jointly by the Mathematics Unit of the Division of Curriculum and Instruction and the Office of Educational Assessment, are intended to help teachers and students derive the optimal value of these tests.

This document includes information and strategies about:

- Test Administration---------- pp. 2-15
- Classroom Activities to Prepare Students for the Tests---------- pp.16-26
- Interpretation of Test Results---------- pp.27-36
- Implications for Staff Development---------- pp.37-38

Excerpts from the 1987 testing manual will be included.

CHARACTERISTICS OF THE SPRING 1987 NEW YORK CITY MATHEMATICS TEST

A. SPECIAL CHARACTERISTICS OF THE PRESENTATION OF CHOICES

1. NG

In the computation part of the test, NG means "not given" and is the last answer choice for all or most questions.

2. Comma Only Used in Five-Digit or Larger Numbers

For example: 20,000
2000
40,000
NG

3. Left Justified Alignment of Answer Choices

Answer choices are printed with left justified alignment.

For example:		
	3	300
	6	0.30
	15	30
	NG	NG

4. Numerical Options in Non-Sequential Order

When there are numerical options as answer choices, they are not listed in increasing or decreasing order.

Example of possible choices: 10, 5, 20, 15
NOT 5, 10, 15, 20

B. ADDITIONAL SPECIFIC CHARACTERISTICS OF THE TESTS BY GRADE

For the Spring, 1987 shelf test, Form M will be used plus a general survey. This survey contains some items selected from other published tests and some new items which are being created to cover instructional objectives which are not on the shelf tests. The general survey for 1987 will be designed specifically for each grade even though grades 3 and 4 will take the same shelf test, grades 5 and 6 will take the same shelf test, and grades 7 and 8 will take the same shelf test. Grade 2, however, will have its own individual test. Not every student will be tested on the same items in the general survey. These survey items will not be included in the students' scores on the 1987 tests.

In Spring, 1988 the test will include some items from the group of survey items tested in Spring, 1987. These items will be included in the students' scores. Each grade will take its own test.

MATHEMATICS

Items in Mathematics for the Intermediate and Advanced 1 levels cover Concepts, Problem Solving, and Computation. The Concepts Test items involve the meaning of large numbers, decimals, and fractions, and some advanced topics like functions and equations, geometry, and customary and metric measurement. Problem Solving items include multi-step word problems. Some require the student to solve the problem; others ask that the number sentence needed for the solution be chosen. Graphs and statistics are also measured. The Computation Test items measure how well students can compute with whole numbers, decimals, fractions, and percents.

Mathematics: Concepts

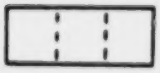
What To Do
Read the sentence. Pick the right answer. Mark the letter for your answer.

1 Which of these figures is divided into thirds?

A

B

C

D 

2 Lucy had 15 sheets of notebook paper. She gave 6 sheets to her friend, Arnold. How many sheets did she have left?

E 8

F 9

G 21.

H 12

Mathematics: Problem Solving

What To Do
Work each problem. Find the right answer or choose the number sentence that you would use to solve the problem. Mark the letter for your answer.

1 Brian has 6 blue pens and 3 green pens. How many pens is that in all?

A $6-3=\square$

B $6-\square=3$

C $6+3=\square$

D $3+\square=6$

Mathematics: Computation

What To Do
Work each problem. Pick the right answer. Mark the letter for your answer. Mark NG if a right answer is NOT GIVEN.

1

$$\begin{array}{r} 14 \\ \times\ 2 \\ \hline \end{array}$$

A 24

B 28

C 38

D NG

2

$$\begin{array}{r} 26 \\ +62 \\ \hline \end{array}$$

E 44

F 83

G 84

H NG

Grade 2

1. Numbers may be printed in all capitalized letters.

   For example: 150 may appear as ONE HUNDRED FIFTY.

2. Reference to days in a monthly calendar is sometimes abbreviated as - S, M, T, W, T, F, S. Therefore, students must be familiar with the sequence of days of the week.

3. For dictated test questions, the suggested timing is approximate, it is not exact. A dictated item may be read once only. Dictated items may appear in both the Concepts and Problem Solving sections.

4. Grade 2 is the only grade that will mark answers in the test booklet. Grades 3 and higher will use separate answer documents.

5. A practice test for students in Grade 2 will be available and will be delivered at the same time as the blank answer documents (approximately mid-March). Schools should use these as soon as possible to familiarize students with the format of the test.

Grades 3 and 4

1. Students use separate answer documents for the first time in Grade 3.

2. For dictated test questions, the suggested timing is approximate, it is not exact. Dictated items may be read only once. Dictated items appear only in the Problem Solving section.

3. Some graphs are smaller in size than graphs on the grade 5/6 test.

4. Students need to know equivalent fractions.

   For example: $\frac{1}{2}$ is the simplest form of $\frac{4}{8}$.

Grades 5,6,7 and 8

1. There are no dictated items at these levels.

2. The correct answer may be given in simplified form.

   For example: $1\frac{1}{6} + 2\frac{1}{6} =$

   Here, the correct answer is $3\frac{1}{3}$, not $3\frac{2}{6}$.

Grades 7 and 8

1. There are no dictated items at these levels.

2. Positive and negative integers may be presented in two ways, with the sign centered and with the sign raised. Students must be able to recognize the item in either form.

   For example: $(-2) + (+5)$ or $^{-}2 + ^{+}5$

   $(-4) - (-1)$ or $^{-}4 - ^{-}1$

TESTING ADMINISTRATION CONSIDERATIONS

Materials Needed for Administering the Metropolitan Achievement Tests:

- o A MAT 6 booklet for each student
- o A General Survey Booklet for each student in grades 3 and above.
- o An answer sheet for each student
- o A MAT 6 booklet (and an answer sheet) for the teacher to use for demonstration during test administration
- o The Directions for Administering
- o A #2 pencil with eraser for each student, plus extras to cover breakage
- o A stopwatch, or watch or clock with a second hand, for timing tests with time limits
- o Scratch paper for each pupil for the Mathematics: Problem Solving and Mathematics: Computation Tests. This should be collected and discarded after testing.
- o A "TESTING - DO NOT DISTURB" sign for the door

Arranging the Testing Room

- o Put a "TESTING - DO NOT DISTURB" sign on the door.
- o Arrange the seating so that students cannot copy from on another.
- o See that all students' desks or tables are clear and there is enough room for the test booklet (and answer sheet, if used).
- o Cover or remove from walls and chalkboard any materials that might help the students in answering questions.
- o The testing room should have sufficient light, ventilation, and freedom from noise. Try to avoid locations or times when other students, school bells, or public address system announcements can interfere with concentration.

During Testing

Time Limits

- o Time limits are necessary to ensure that comparisons to the norming population can be made. They assure that testing moves along at a reasonable pace and can be scheduled accurately.
- o Time limits for the test should be strictly followed. A watch with a second hand or a stopwatch is necessary to assure accurate timing.
- o Tests that are teacher dictated are not subject to precise time limits. However, they should be administered in approximately the time indicated.
- o Testing may be stopped early if all students have finished before the stated time limit. Be sure all students are satisfied that they have completed the

task.

- Students who have completed the test ahead of time, or who have abandoned the task as beyond them, should maintain the quiet testing atmosphere so that others can continue working.
- Special boxes for recording the times at which students start and are supposed to stop are provided in appropriate places throughout the directions.

Directions and Samples

- Read the directions to the students exactly as they are written.
- Read with a natural tone of voice and in a natural manner.
- Be sure students know what they are to do before starting a test. After directions are read verbatim, they may be further clarified if students do not understand.
- Sample items may be reproduced on the chalkboard. They may be discussed or explained in detail. Do not make up additional sample items.
- Dictated items should be read only once unless otherwise indicated. If some disturbance occurs during the reading that prevents students from hearing clearly, the item may be repeated.

Proctoring the Test

- Move around the room after the test has started to make sure that everyone is working in the correct place, but do not hover over a student.
- Be sure that everyone is following instructions. Additional instructions may be given to students who seem confused, but do not give help on specific questions.
- The teacher may, with discretion, give additional encouragement to students who have become discouraged and slowed down or stopped trying. Urge on those who dawdle.
- Prevent students from helping one another. This affects the true picture of each student's achievement and disturbs other students.
- If a student forgets how to change an answer, remind him or her to mark the new answer and then erase the first mark completely.

Special Considerations

- In order to ensure that all students are tested under standardized conditions, the use of calculators, dictionaries, or similar aids is not permitted.
- The amount of time allowed for the test is not announced to students at lower levels. However, if students inquire, they may be told.
- Do not give a direct answer when a student asks if his or her answer to a particular question is correct. Use some neutral response like, "Mark the one you think is correct and go right on."
- Although random marking is to be discouraged, students should mark all items for which they can select a reasoned response. They should be urged to mark a response to any item they think they know. Do not use the term "guessing," which frequently implies random marking.
- Do not allow students to lose testing time by being overly meticulous about how they mark their answer, even though the marks in machine-scorable booklets and on answer documents should fill the answer circle.
- A note should be made of all disturbances or irregularities that occur during testing.

I PROVIDE PRACTICE WITH MULTIPLE CHOICE FORMAT

Multiple choice items similar to those on the NYC Mathematics Test should be used in the instructional program for classroom practice in the warmup, applications and homework segments of the lesson and for inclusion in teacher-written class tests. While there is a considerable benefit to students in coping with a collection of teacher-made test items under test conditions, this kind of practice can easily be overdone. This kind of practice will have its greatest value if it is preceded by a discussion of the scope and form of the test, tips for taking the test, and student attitudes about the test, and followed by a comprehensive review of the solutions of all of the test items. Guidelines for the kinds of instructional activities that should precede the "practice test" experience may be found in the section about decreasing test anxiety.

Multiple-choice items may also be used integrated in the regular program of instruction. The following practices are recommended:

In the warmup, it will usually be more effective to present the item without the choices. Students may be challenged to estimate the result before performing the calculations.

Teacher-made multiple choice items should include the "NG" (not given) option, both as correct and incorrect responses.

Teacher-Created Multiple Choice Questions

Creating multiple choice questions with appropriate distractors by the teacher can be helpful in analyzing patterns in student errors as well as giving students practice to increase their test sophistication. Pupils' choices of particular distractors can provide insight into the errors pupils are making so remediation can take place.

1. What errors are made by students who choose a or c below?

$$\begin{array}{r} 24 \\ \times\ 5 \\ \hline \end{array}$$

a) 100 c) 29
b) 120 d) NG

2. What distractors, along with the correct response, would be helpful in diagnosing pupil difficulties and common errors in the examples below?

$5\overline{)74}$

a) c)
b) d)

$$\begin{array}{r} 1.3 \\ \times 0.2 \\ \hline \end{array}$$

e) g)
f) h)

What comes next to complete the pattern?

4, 7, 10, 13, ☐

a) c)
b) d)

$\frac{1}{3} + \frac{1}{4} = \square$

e) g)
f) h)

II PROVIDE PRACTICE WITH DICTATED ITEMS

In Grades 2 and 3/4, a portion of the test is dictated to the students. To familiarize the students with this format, opportunities for dictation practice should be found.

Many types of items can be dictated for practice. Some of these are:

Dictation of Numbers

- five hundred three
- ninety-one thousand, four hundred sixty-two

Dictation of Examples

- two hundred seventeen plus one hundred five
- forty-nine divided by five
- one and two tenths minus nine tenths

Dictation of Problems

- Ted had 9¢. He spend 4¢. How much did he have left?

Dictation in other subject areas will also help students improve listening skills as well as offer practice in transcribing information from oral to written from.

III USE VARIETY OF LANGUAGE IN QUESTIONING

Many times students are accustomed to hearing or reading a question posed in only one way. Varying the way a question is worded offers the students an opportunity to develop a better understanding of concepts as well as sophistication with regard to recognizing the concept presented in a variety of ways.

1. a) What is the quotient when 287 is divided by 7?
   b) How many groups of 7 can be made from 287 items?
   c) How many sevens are in two hundred eighty-seven?
   d) What quotient will be produced from a dividend of 287 and a divisor of 7?

2. In what other ways can these questions be stated?

   a) How much is 30 times 20?
   b) What is the value of the digit 2 in 2,436,197?
   c) How much is 298 and 418?
   d) What is left if we subtract $\frac{2}{3}$ from $2\frac{5}{8}$?

Students should also be challenged to find new ways to restate problems which they are given.

IV ENCOURAGE STUDENTS TO COMPOSE QUESTIONS

Asking students to compose their own questions not only motivates them and their classmates to solve the problem they have created, but also gives the teacher insight into how well the student understands the concept, operation, or process related to his/her question. Charts and graphs, as well as many other forms of data, lend themselves readily to student developed questions. Not only may students compose new questions about a given graph, but they may also be asked to make or find similar graphs and write questions about these, too.

The teacher should initiate the process by providing a model question and eliciting some questions from the group before asking individual student to compose original questions.

What question might a student create regarding the information on the chart below?

BAKERY PRICE LIST

Layer Cakes	$4.95	Wheat Bread	$1.15
Cheese Cakes	5.25	Rolls	.35
Donuts	.29	White Bread	1.15
Cupcakes	.39	Rye Bread	1.25

Composing problems similar to ones they are given helps students internalize their learning and assists transfer of information to long term memory. These student-written problems can be added to the pool of questions for future class practice, homework, and tests.

V DISCUSS QUESTIONS THAT ASSUME KNOWLEDGE

Questions relating to factual information we assume the student knows will provide practice for the student who has this knowledge and will help make other students aware of these facts. Certain numbers bring information to mind immediately, e.g., 12 refers to the number of items in a dozen as well as the number of months in a year. We expect students to know facts such as these.

What knowledge is assumed in the following questions?

1. How many eggs are in five dozen?
2. Mrs. Kramer can type 80 words per minute. How many words can she type in 1 hour?
3. How many inches are in three yards?
4. A recipe calls for 8 cups of milk. How many quarts of milk are needed?
5. The temperature is 45°F. How many degrees must the temperature drop before the water in the pond will start to freeze?

VI HIGHLIGHT PROBLEM SOLVING STRATEGIES

In reviewing solutions to any problems, teachers should help students use these strategies to clarify their undestanding:

1. Rephrase the problem in your own words.
2. Make a diagram of the situation if possible.
3. Estimate the answer and eliminate unreasonable choices.

An expanded list of problem solving strategies appears in the "To the Student" notes. Review the practice questions for grades 2 and 3-8. Can any choices be eliminated because they are unreasonable?

VII SPIRAL PRACTICE IN THE HOMEWORK

Homework reinforces what is learned in class and aids the teacher in identifying parts of a lesson which pupils did not understand. To assure retention of previously learned material, it should be cyclical in nature.

Mathematics instruction is most effective when students find common sense in what they learn. Research shows that learning based on rote memorization is sometimes more "effective" over a short interval, but this advantage diminishes as the instructional period lengthens. Learning based on discussion of meaning is more effective for long term retention of concepts.

Although drill is not a preferred way of learning mathematics, it is an excellent way to consolidate mathematical learning which has been developed on a teach-for-understanding basis. Practice that involves retrieval of skills learned several days earlier is more effective in transferring knowledge to long-term memory than practice that follows immediately on the heels of learning a specific skill.

Homework should be planned well in advance and designed to include a variety of topics, reviewing prior and current learning and laying a base for future discoveries. Less than 50% of an assignment should relate to work learned in the past week. Review topics should be included in a "cyclical" pattern, recurring periodically to keep prior learning fresh.

Grade ___

	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
Component 1 (Current Work)	Unit ___	Unit ___	Unit ___	Unit ___	Unit ___
Component 2 (Recent Past)	(Unit __)	(Unit __)	(Unit __)	(Unit __)	(Unit __)
Component 3 (Distant Past)	(Unit __)	(Unit __)	(Unit __)	(Unit __)	(Unit __)
Component 4 (Preparation for Upcoming Lessons)	(Unit __)	(Unit __)	(Unit __)	(Unit __)	(Unit __)

Rationale for Component 4

Prepare students with skills needed for coming units:

Unit ___ :

Unit ___ :

Unit ___ :

Unit ___ :

Grade 4

	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
Component 1 (Current Work)	Unit 38 W* subt. 5-digit nos. with exchange and check; solve word problems	Unit 38 W subt. from 10,000 and multiples of 10,000 and check; solve word problems	Unit 39 identify improper fractions and mixed nos.; change improper fractions to whole nos. or mixed nos.	Unit 39 change mixed no. or whole no. to improper fraction	Unit 40 add and subt. mixed nos. with common denominators
Component 2 (Recent Past)	write tenths and hundredths as decimals (Unit 31)	compare and order two or more decimals (Unit 32)	identify prime and composite nos. (Unit 34)	identify radius, chords, diameter of a circle (Unit 35)	subt. 5-digit nos. with exchange and check (Unit 38)
Component 3 (Distant Past)	round 4- and 5-digit nos. to nearest thousand (Unit 8)	solve open sentences (Unit 15)	W solve 2-step word problems (Unit 16)	W multiplication facts for 11 and 12; solve word problems (Unit 18)	W Problem solving: Guess and Check (Unit 19)
Component 4 (Preparation for Upcoming Lessons)	add and subt. fractions with common denominators (Prepare for Unit 40) (Unit 14)	W solve problems related to add. and subt. of fractions with common denominators (Prepare for Un. 42) (Unit 14)	identify perpendicular and parallel lines (Prepare for Unit 42) (Unit 30)	divide 2-digit nos. by 1-digit nos., no regrouping (Prepare for Units 44-45) (Unit 22)	divide 2-digit nos. by 1-digit nos., with regrouping (Prepare for Units 44-45) (Unit 22)

Rationale for Component 4

Prepare students with skills needed for coming units:

Unit 40 : Addition and Subtraction of Mixed Numbers with Common Denominators

Unit 42 : Parallelograms

Unit 44 : Distributive Property of Division

Unit 45 : Division of Money Values

* W = Word Problems

Grade 4

	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
Component 1 (Current Work)	Unit 38 *R subt. 5-digit nos. with exchange and check; solve word problems p. 291 #1,2,3,11	Unit 38 W subt. from 10,000 and multiples of 10,000 and check; solve word problems p. 293 #1,2,3,9	Unit 39 identify improper fractions and mixed nos.; change improper fractions to whole nos. or mixed nos. p. 189 #1-4	Unit 39 change mixed no. or whole no. to improper fraction p. 195 #1-4	Unit 40 add and subt. mixed nos. with common denominators p. 186 #1-4
Component 2 (Recent Past)	write tenths and hundredths as decimals p. 245 #7,8 (Unit 31)	compare and order two or more decimals p. 248 #12,13 (Unit 32)	identify prime and composite nos. p. 256 #3-5 (Unit 34)	identify radius, chords, diameter of a circle p. 78 #5,6 (Unit 35)	subt. 5-digit nos. with exchange and check p. 291 #4,5,6 (Unit 38)
Component 3 (Distant Past)	round 4- and 5-digit nos. to nearest thousand p. 37 #9,10 (Unit 8)	solve open sentences p. 49 #22,23 (Unit 15)	Solve 2-step word problems W p. 199 #7 (Unit 16)	multiplication facts for 11 and 12; solve word problems W p. 34 #15-17 (Unit 18)	Problem solving: Guess and Check W p. 99 #4 (Unit 19)
Component 4 (Preparation for Upcoming Lessons)	add and subt. fractions with common denominators p. 142 #7,8,9,15 (Prepare for Unit 40) (Unit 14)	solve problems related to add. and subt. of fractions with common denominators W p. 142 #20,21 (Prepare for Unit 40) (Unit 16)	identify perpendicular and parallel lines p. 150 #6-8 (Prepare for Unit 42) (Unit 30)	divide 2-digit nos. by 1-digit nos.; no regrouping p. 104 #21,22 (Prepare for Units 44-45) (Unit 22)	divide 3-digit nos. by 1-digit nos., with regrouping p. 106 #19,20 (Prepare for Units 44-45) (Unit 22)

Rationale for Component 4

Prepare students with skills needed for coming units:

Unit 40 : Addition and Subtraction of Mixed Numbers with Common Denominators

Unit 42 : Parallelograms

Unit 44 : Distributive Property of Division

Unit 45 : Division of Money Values

* W : Word Problems

INSTRUCTIONAL STRATEGIES FOR REDUCING TEST ANXIETY AND INCREASING TEST WISENESS

For some students a testing situation is an anxiety-provoking experience. To reduce stress and anxiety, help students see that test-taking is part of the ongoing self-assessment process. Our goal should be to guide students to be objective in evaluating their strengths and weaknesses, to develop a greater desire for self-improvement, and to become more realistic in decision-makng. The teacher should help students to understand that tests serve as one tool for assessing strengths and weaknesses and that such assessment is useful in planning for improving their skills. Success is not a pure function of luck but can be attained by changes in their own attitudes and behaviors.

To help students obtain the greatest benefit from their school experience, the teacher must be aware of their emotional as well as their intellectual needs. The student must feel that he or she belongs, is accepted, can be himself or herself, will get a fair chance, and that someone will listen. During test-taking, provide a climate in which students are not exposed to unnecessary interruptions and noises. Before the test, creating an orderly and quiet environment will put them at ease. Timing is important, since students may feel overly excited after a gym period or be less alert at certain hours of the day; help students become aware of how these factors influence their performance. Use pre- and post-test class discussion to familiarize students with the testing process. Students thrive on approval and wither under criticism. They should be praised for effort as well as performance, keeping in mind that any praise has to be sincere. Trying hard shows responsibility, even if the results are not perfect. Emphasis should be given to strengths and the development of specific strategies for overcoming weaknesses.

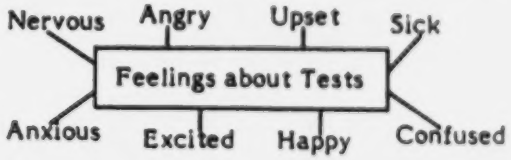
In essence, you should create a trusting and success-oriented environment throughout the school year. To achieve improved performance, students should:

- develop positive self-images as learners, risk-takers, and test-takers
- be aware that the role of a test is to measure their subject matter strengths and weaknesses
- develop a sense of responsibility for setting realistic goals for their own self-improvement

The following strategies can assist in achieving these objectives:

1. Help students develop positive self-images as test-takers by leading the class in a discussion of their perceptions and feelings about taking a test.

   Have students brainstorm to develop a list of feelings they have about testing. The teacher may wish to use semantic mapping or webbing techniques, asking, "What words come to mind which describe your feelings about taking a test?" and diagramming the students' responses in this matter:



Another approach would be to offer the following open-ended sentences as a basis for discussion or as a writing assignment followed by discussion:

A test is like a _____.
When I take a test I feel _____.
When I'm prepared for a test I'm _____.
When a test is easy it makes me feel _____.

Elicit responses and discuss reasons for each response by asking, "Why do you think you feel this way?" Emphasize that negative feelings toward a task usually increase anxiety, which in turn affects performance. Explain that by listening to and sharing feelings about testing, students will reduce their own anxiety.

Assist students in dealing with and changing negative attitudes about testing. For example, ask students to describe a task they hate to do. After a negative statement, ask students to make a positive comment about the same task.

Negative: I don't like to clean my room.
Positive: When I clean my room, it is easier to find the things I need.

Negative: I'm not happy when I take a test.
Positive: After I take a test, I can see what I understand well and what I still need to work on.

2. Help to create and emphasize success by asking students to regularly face the question, "What have I learned today?" For example, after completing an activity, ask students to write:

Today I learned about (or how to) _____.
The most important idea that I learned today is: _____.
Something I learned today about thinking is: _____.
A way that I will use these ideas is: _____.
A question I have about what I am learning is: _____.

Processing new learning in this way will help students clarify their understanding of new concepts, and help the teacher identify areas in special need of review and reinforcement. It will also help the students focus on the task rather than on their anxiety, and will help them appreciate their own progress. Provide positive feedback, stressing that each student has improved or learned something new.

3. Share or develop with your students general guidelines for preparing for examinations and taking examinations, such as those on the following pages. Be sure to emphasize the importance of reviewing answers, checking by estimation, and asking, "Is this answer reasonable?" as well as checking by using an alternative method to find the result and detecting errors by repeating calculations.

TO THE STUDENT: HOW TO PREPARE FOR THE EXAMINATION

The best way to prepare for an examination is to do your best all year on your regular school work. If you have done so, much of the preparation for the examination is behind you because the examination is based on what you have been taught during the year.

You can improve your chances of doing well on the examination by becoming familiar with the general scope of the test as well as with the specific ways in which the information is asked for. Practice in answering the sample questions provided should help you to become familiar with general test content with the various types of questions. However, in the several weeks preceding the examination, you should concentrate on your work in the areas covered by the test. You should aim for general improvement in an area where you know you are weak.

Mathematics. The questions on the mathematics test are based on the school curriculum and measure the knowledge and skills that you have acquired during your school years. Although many questions measure just your computational ability, most questions are designed to measure your ability to reason mathematically. These questions tap your understanding of mathematical concepts, your ability to solve word problems, and your ability to apply your mathematical knowledge in a new situation. It will help if you develop habits of carefulness, neatness, and accuracy. Practice in doing sample exercises may also increase your speed. All of these will contribute to a better test score.

TO THE STUDENT: PROBLEM SOLVING STRATEGIES

Many students find these techniques useful when facing a new problem. If you can form the habit of using these approaches, you will find you are more successful in solving problems. It is a good idea to look for more than one way to solve each problem; this will help you confirm whether the first solution was correct, and also help you discover more efficient and powerful methods.

1. Restate the problem in your own words. State clearly what you are trying to find and what information is given.
2. Make a diagram of the situation if possible.
3. Estimate the answer. Eliminate unreasonable answers.
4. Make up a problem which is similar but simpler. Solve that one first, then imitate your procedure in the given problem.
5. Use analogy.
6. Keep good records (perhaps using a chart or graph).
7. Look for patterns.
8. Use symmetry.
9. Divide the situation into parts. Handle each separately, then combine the results.
10. Make an educated guess. Examine the consequences of this value (Does it fit the given situation?) and adjust the guessed value until it matches all the conditions of the problem.
11. Work backward.

TO THE STUDENT: TIPS FOR TAKING THE TEST

1. Be alert. A good night's sleep the night before the examination will be more helpful than trying to cram many years of study into a few hours. Tired students tend to make careless mistakes.

2. Be calm and confident. If you feel tense during the examination, take a minute to relax. Don't worry too much about any one question. Other students are finding some of the questions difficult too. Remember, no one is expected to answer all the questions correctly.

3. Wear comfortable clothes so you will feel at ease.

4. Read carefully all directions printed in the test booklet and listen carefully to directions given to you by the examiner.

5. Watch your time and adjust your speed if necessary. Don't spend too much time on any one question. If you don't know the answer, go on to the next question. Be sure to answer all the questions that you know the answers to. If time permits you can go back and work on the questions that you skipped or were not sure of.

6. Don't be careless. If you have extra time check your work to make sure you made no foolish mistakes.

7. If you are not entirely sure of the answer to a question, you may mark it anyway. Wild guesses may not help much but if you can eliminate one or two of the possible answers, you should mark your best choice. This is usually better than skipping the question entirely.

8. If scratch paper is provided, use it, as it may help you to work your way through a problem on paper. (Do not bring any books, notes or papers of your own into the examination.)

9. Be careful both in deciding upon the answer to each question and in recording your choice on the answer sheet. Be sure to mark the answer to a question in the row of the answer sheet that has the same number as the question you are working on. Mark only <u>one</u> answer to a question. If you mark more than one answer, the question will be counted as wrong.

10. Be sure to have extra sharpened pencils and an eraser so that if a pencil breaks or if you want to change an answer, you won't waste testing time looking for what you need in order to continue.

Interpretation of Citywide Mathematics Test Scores and Test Score Reports

The Office of Educational Assessment (O.E.A.) produces a variety of reports regarding students' performance on the spring, citywide mathematics test. Reports generated include:

1. An individual student report (letter to parents)
2. An individual student label
3. Class rosters
4. School summary reports
5. District summary reports
6. School objective summary reports
7. District objective summary reports

Sample copies of these reports are attached. A brief description of the information included in these reports and an explanation of how to interpret the information follows.

1. Individual student report (letter to parents)

This report is sent to the schools for transmittal to parents. The letter describes the test briefly and reports how the student performed in terms of percentile rank.

The first "ruler" shows with an arrow the child's percentile rank on the total test. That is, it shows that the student did as well as or better than a certain percent of students in a nationally representative group. An additional arrow indicates that the 50th percentile is the midpoint, or the average score for all students. This is provided so that parents can see in general whether their children performed above or below the average.

The second "ruler" shows the student's percentile rank on each of the three subtests that make up the test: concepts, problem solving, and computation. This ruler also indicates the average or "medium" score (50) and the lowest (1) and highest (99) scores possible. The individual student's percentile rank on each of these subtests is printed at appropriate points on the ruler.

2. Individual student label

This label is sent to schools to be placed on student's permanent record cards. The label includes on line 1 student name and 9-digit I.D. number; on line 2 student date of birth, sex, borough, district, school, grade, and class; on line 3 test, form, level, and date of testing; and on lines 4 and 5 test scores: ((RS) raw score, (SS) scaled score, (NCE) normal curve equivalent, (NP) national percentile or percentile rank, and (ST) stanine). Line 5 also may include "status" information, e.g., exempt LEP or absent.

The raw score is the number of items on the test answered correctly by the student.

The scaled score is a score usually used for analytic purposes and is particularly suitable for analyzing growth across forms or levels of a test. This type of score is most meaningful to evaluators; school level personnel will rarely find it useful.

The normal curve equivalent is similar to the scaled score in that it is a score used primarily by evaluators of funded programs for analytic and reporting purposes. It is based on a child's percentile rank, but is not inherently meaningful for school level personnel.

The national percentile (or percentile rank) is the type of score that school staff and parents should find most meaningful. These scores range from 1 to 99; 50 is the middle score. A national percentile indicates the percentage of students in the national norming population that received scores equal to or less than a particular score. That is, students who receive a particular percentile rank perform as well as or better than that percent of the national group.

National percentile scores are the primary scores used in New York City for reporting to parents, etc. A percentile rank of at least 50 is used to designate students "at or above grade level." Percentile ranks at other points are used as promotional gates criteria, Chapter I eligibility criteria, etc.

The stanine is a normalized standard score of nine units, 1-9. Stanines of 1-3 are low; 4-6 average, and 7-9 high.

3. Class rosters

These reports are given to you by borough, district, school, grade, and class. Students are listed alphabetically within class. Both schools and districts receive copies of these reports.

These rosters list students by reference number, I.D. number, and name; sex and birthdate; "not tested" status (e.g., absent, LEP or S.E.); and "tested" status" category (Special Education or Resource Room).

The rosters then provide test score information (raw scores, scaled scores, normal curve equivalents, national percentiles, and stanines) for each subtest and for the total test.

4. School summary reports

These reports provide summary information by grade for each school. For each grade, the number of answer documents submitted is reported; this number is then broken down by number tested, absent, excused, "no scores", and LEP-exempt.

The top half of the report gives the number and percentage of students falling in each of the four quartiles. Quartiles are those points that break the national distribution into four equal groups. The first quartile includes percentile ranks of 1-25; the second, 26-49; the third, 50-75; and the fourth, 76-100. A grade that approximates the national distribution would have about 25 percent of its students in each quartile. The number and percent of students in the top two quartiles adds up to the number and percent of students "at or above grade level."

The bottom half of the report gives mean raw scores, NCE's, and percentile ranks on the subtest and total test. These mean scores are based on tested students only.

5. District summary report

These reports give the same information as the school summary reports just described, except that they are bsed on the total number of students by grade in the district, rather than the school.

6. School objective summary report

This report indicates, in general, the number and percent of students in the school scoring "low", "average", or "high" on objectives, clusters, and subtests. The number of items needed to classify performance as low, average, or high is supplied by the publisher and is based on national norms.

The report also provides the percent of students in the district receiving low, average, or high scores, so that the school can compare its performance to the entire district.

This report provides the most detailed instructional level information of the reports described. It provides information on how groups of students performed on objectives, rather than on entire subtests.

7. District objective summary report

This report provides the same type of information as the school report just described, except that it provides district level information and citywide information as a comparison.

New York City Board of Education

Office of Educational Assessment
110 Livingston Street, Brooklyn, New York 11201

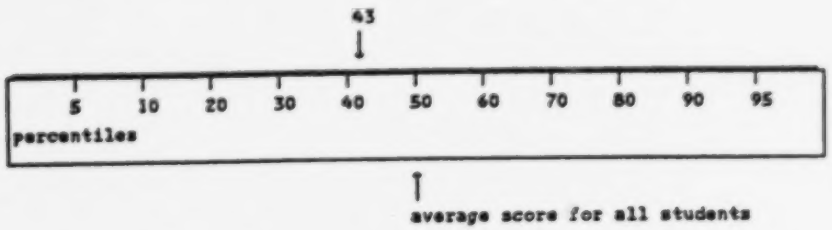
BORO/DISTRICT: SCHOOL: GRADE/CLASS: 0 - DATE: 05/23/86

TO THE PARENTS OR GUARDIANS OF

In Spring, 1986, we gave a new, more up-to-date citywide mathematics test known as the Metropolitan Achievement Test (MAT). The MAT helps us answer the question: how well is your child learning mathematics? Of course, this test is only one way of knowing how students are doing in school.

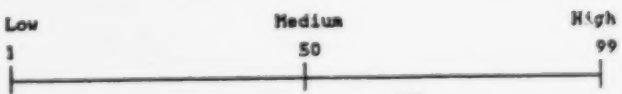
When we gave the test, we asked your child to use a number of mathematics skills to answer 84 questions. Your child correctly answered 52 questions.

In terms of percentile rank, your child scored:



This means that your child scored better than 43 percent of 4th graders across the nation.

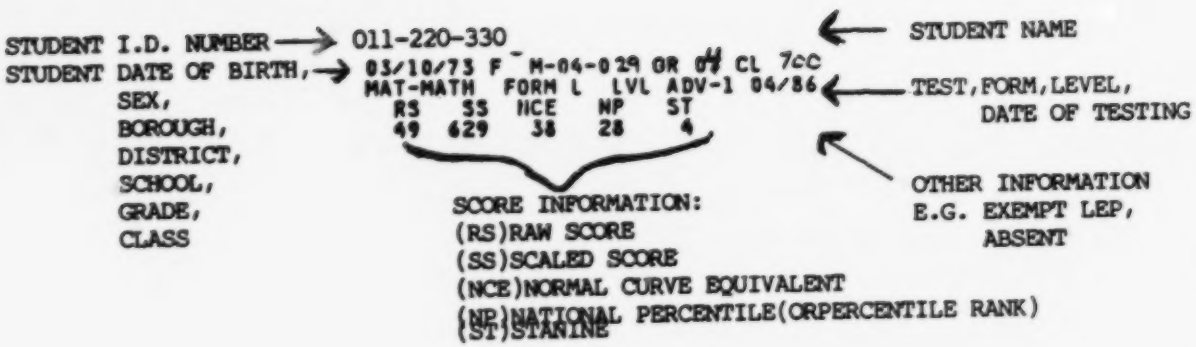
The test also gave us information about how well your child uses a variety of basic mathematics skills. We tested three types of skills. Here are your child's percentile scores in each of the three skill areas:



Concepts	66
Problem Solving	20
Computation	48

Your child's teacher can give you more information about the test and about your child's progress.

INDIVIDUAL STUDENT LABEL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03/10/73 F M-04-029 GR 04 CL 7CC
MAT-MATH FORM L LVL ADV-1 04/86
RS SS NCE NP ST
49 629 38 28 4
STUDENT NAME
TEST, FORM, LEVEL, DATE OF TESTING
OTHER INFORMATION E.G. EXEMPT LEP, ABSENT
SCORE INFORMATION:
(RS)RAW SCORE
(SS)SCALED SCORE
(NCE)NORMAL CURVE EQUIVALENT
(NP)NATIONAL PERCENTILE(ORPERCENTILE RANK)
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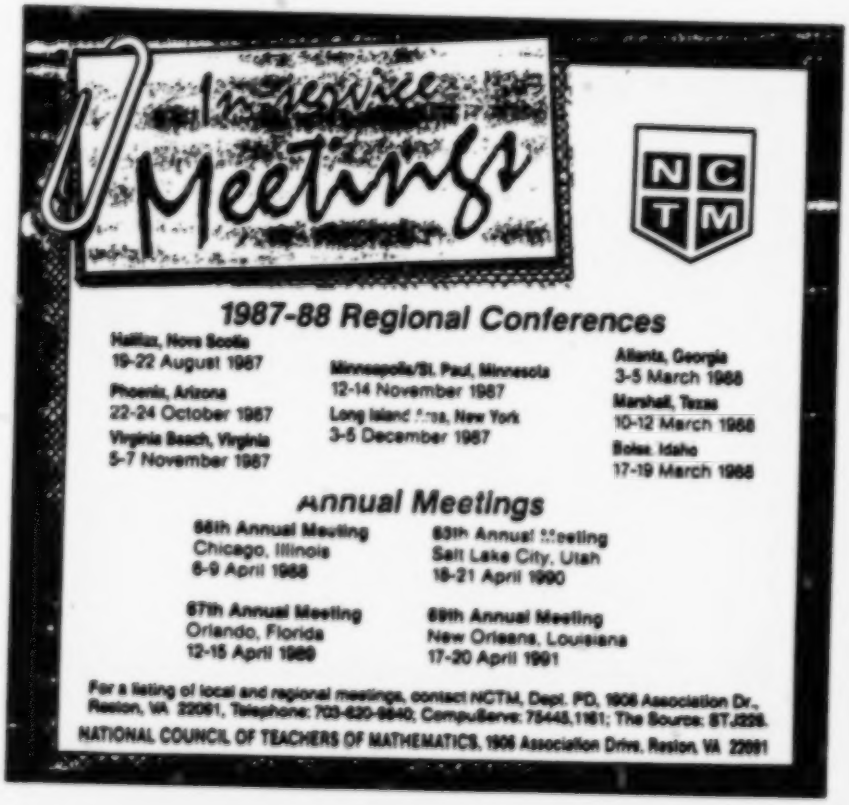

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WHAT IS EFFECTIVE TEACHING?

"RESEARCH SAYS..."

1. Discussion of Meaning

Mathematics instruction is most effective when students find common sense in what they learn. Research shows that learning based on rote memorization is sometimes more "effective" over a short interval, but this advantage diminishes as the instructional period lengthens. Learning based on discussion of meaning is more effective for long term retention of concepts.

Although drill is not a preferred way of learning mathematics, it is an excellent way to consolidate mathematical learning which has been developed on a teach-for-understanding basis. Practice that involves retrieval of skills learned several days earlier is more effective in transferring knowledge to long-term memory than practice that follows immediately on the heels of learning a specific skill.

2. Verbalization

Students who were asked to verbalize what strategies they had used to solve a set of problems were more successful on a subsequent set of problems than students who had not been asked to put their strategies into words.

3. Questioning

Effective teachers of mathematics:

- ask more questions and lecture less frequently
- ask more questions that provoke higher-level thought processes (analysis, synthesis)
- ask more follow-up questions after a student response
- provide more wait-time for answers to questions
- encourage students to ask questions
- train students to ask effective questions

4. **High Student Participation in Learning**

Effective teachers try to keep most of their students from slipping into passive learning. The higher the number of student-initiated questions and comments, the higher was student achievement in mathematics.

5. **Activity-Based Learning**

Programs in which the student is actively involved in doing or in seeing something done are most successful. Activity-based instruction may include real-world experiences, working within groups to accomplish a task, multimedia approaches and learning (interest) centers. The activity helps the students connect the mathematical idea to the real world.

Lessons involving manipulative materials will produce greater mathematical achievement than will lessons in which manipulative materials are not used. The instructional purpose of the activity must be clear to the teacher and the students.

6. **Modeling and Practice of Reasoning and Language Skills**

There is a clear correlation between the frequency of mathematics teachers' use of conditional reasoning (e.g., "if... then...") in their speech and their students' understanding of logic.

Students given more opportunities to use logic and language attain greater mastery.

The use of counter-examples of mathematical statements in conjunction with (positive-) examples of statements resulted in higher achievement than did the use of examples alone.

7. **Wise Apportionment of Instructional Time**

More effective teachers use about three fourths of each period for demonstration and discussion, and only about one fourth of each period for individual seatwork.

8. **Homework Integrated with Instruction**

More effective teachers assign homework and check homework regularly. Providing only answers to the previous day's assignment is less effective than reviewing and analyzing the solutions.

Students should spend more than half of their time on tasks they can carry out with high success.

9. **Monitoring Student Performance**

When monitoring of students by teachers is high, students are more effectively kept engaged in their assigned tasks, which leads to higher achievement.

Teachers should circulate about the room while students are working, frequently review assignments and notebooks, evaluate performance through a regular testing program, and elicit articulation of student thinking by asking questions ("How did you get that result?") and by buiding students to compose effective questions of their own about what they are doing.

10. **Careful Analysis of Error**

One important aspect of a teacher's job is to determine how a student goes about getting the answer. When errors are made, careful analysis (through observation and interviews with individual students) may reveal patterns of misunderstanding that are "logical" to the student.

11. **Remediation**

Remedial instruction must be different from previous instruction in the same material. Repeating instruction in the same manner given previously will not be effective. Multisensory experiences are important for both motivational and developmental reasons. Students should be gradually guided toward a concrete, intuitive understanding of mathematical principles, toward the ability to express their understanding verbally and symbolically, and toward the capacity to apply their understanding constructively to unfamiliar situations.

12. **Classroom Climate and Motivation**

The teacher's potential for affecting student motivation, positively or negatively, is awesome. While, in the end, motivation must come from within each student, it can only come when the student feels the excitement of learning, believes his or her efforts are appreciated, forms clear goals, makes some connection between the work done in the mathematics class and those goals, and feels the confidence and freedom to risk attaining them.

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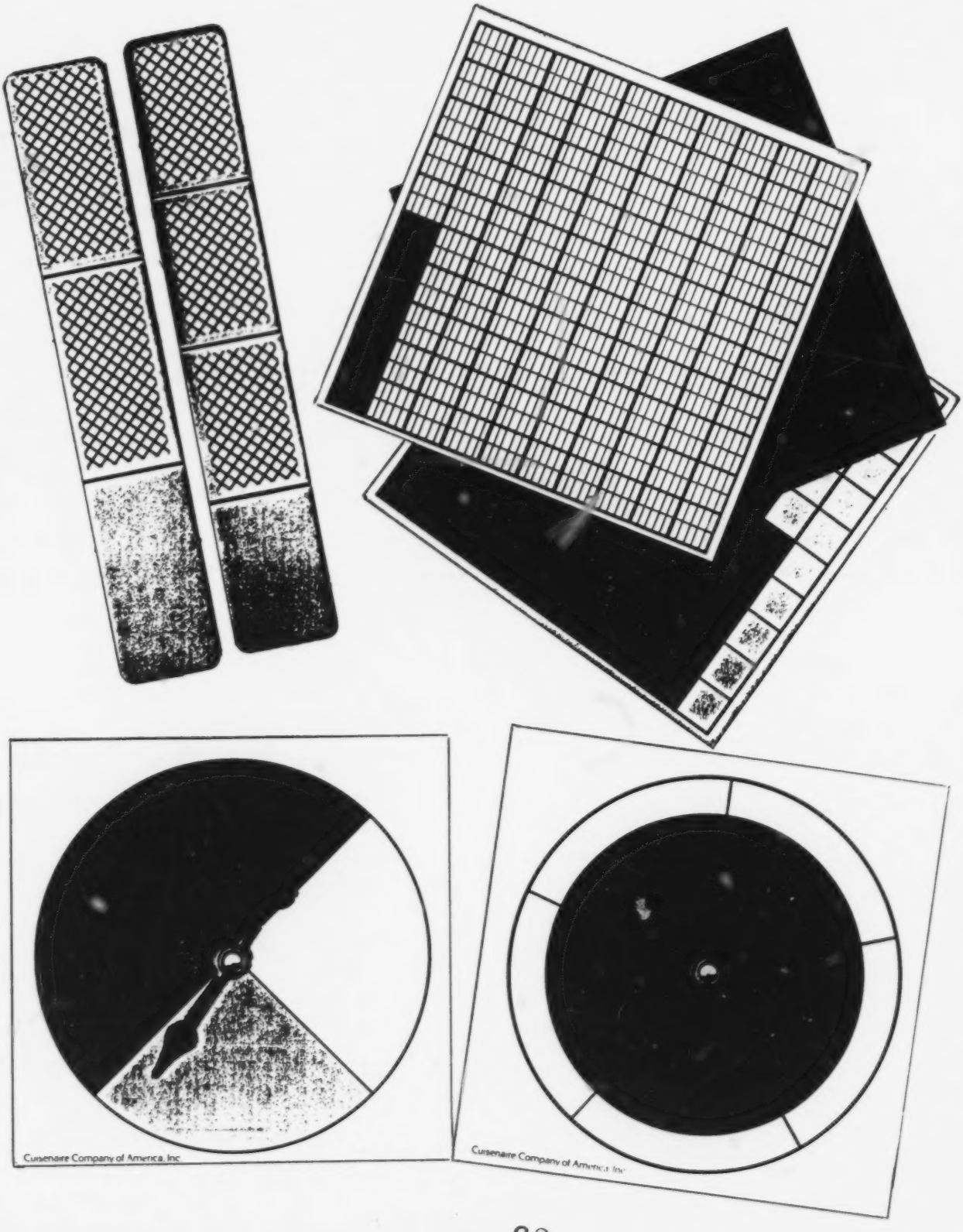
RATIONALE FOR THE USE OF MANIPULATIVE MATERIALS FOR CLASSROOM INSTRUCTION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MANIPULATIVES

Research Findings:

Lessons involving manipulative materials will produce greater mathematical achievement than will lessons in which manipulative materials are not used if the manipulative materials are used well. Manipulative materials should be used frequently in a total mathematics program in a way consistent with the goals of that program. The manipulatives should be used in conjunction with other aids, including diagrams, textbooks, films, and similar materials. The simplest possible manipulative material (i.e. egg cartons or counters) should be used in direct relationship to the mathematical attribute being studied. Manipulative materials should aid in orgainizing content. Through their use, the student should be aided in recording results symbolically.

Questions to Consider:

Do I have a clear understanding of both the activity and its instructional purpose?

Do I actively participate during these sessions using manipulatives by assisting students who require it?

Do I integrate the use of manipulatives frequently in my classroom?

Do I demonstrate a new activity with the whole class?

A Rationale

By **Leonard M. Kennedy**

Many teachers believe very strongly in the benefits of manipulative materials in building a foundation for children's mathematical concepts: "I am firmly convinced that manipulative materials improve children's understanding of the mathematics they learn. My classroom is well stocked with all sorts of commercial and teacher-made materials. My children and I use them nearly every day." Other teachers view them with skepticism and are less inclined to use them.

Classroom teachers see the value of manipulatives once they have used them with their classes.

"There just isn't time. And I'm really not convinced that they are important, anyway. They are a lot of bother." Yet strong evidence supports the use of manipulative materials to help children build a firm understanding of mathematical concepts.

First, what are manipulative materials? Manipulative materials are objects that appeal to several senses and that can be touched, moved about, rearranged, and otherwise handled by children. Young (1983, 12) defines these materials as "objects which represent mathematical ideas that can be abstracted through physical involvement with the objects." They can be objects from the environment, such as money or measuring instruments, or materials specifically designed to teach mathematical concepts, such as base-ten blocks and balances.

Support from Learning Theorists

Research on learning provides information that teachers, curriculum planners, and textbook writers use to develop mathematics programs. The mental-discipline and stimulus-response theories of the nineteenth and early twentieth centuries gave way to meaning theory, espoused by William Brownell in the 1930s. This theory is based on the belief that children must understand the basic concepts that underlie what they are learning if learning is to be permanent. Brownell's discussion of learning generated interest in having children use manipulative materials to form the concepts necessary in learning mathematics.

More recent studies of human learning by Jean Piaget (1952) and Richard Skemp (1982) led them to conclude that all individuals pass through stages as they mature. Piaget believed that cognitive development occurs in four stages. Manipulative materials are significant learning aids in all four stages. Students' mental images and abstract ideas are based on their experiences. Hence, students who see and manipulate a variety of objects have clearer mental images and can represent abstract ideas more completely than those whose experiences are meager. Skemp (1982) believes that a two-level director system governs learning. At the first level the manipulation of objects in and out of the school environments provides the learner with physical activities that form the basis for further learning and the internalization of ideas. At the second level, activities are built on those of the first and are mental rather than physical.

Dienes (1960) advocates the use of manipulative materials by children. He describes the use of multibase blocks and other materials to build children's understanding of numbers. Each of the devices supplies a proper concrete representation of a concept. He advocates the use of "multiple embodiments" rather than a single representation of a concept. The multiple representation of place value occurs when children use loose and bundled sticks, bean sticks, ten frames, abacuses, and other place-value materials to learn the meaning of the Hindu-Arabic numeration system.

The strength of the arguments of these and other theorists for using manipulative materials is persuasive. Learning theories suggest that children whose mathematical learning is firmly grounded in manipulative experiences will be more likely to bridge the gap between the world in which they live and the abstract world of mathematics. The manipulatives help children understand both the meanings of mathematical ideas and the applications of these ideas to real-world situations.

Leonard Kennedy is a professor of education working with elementary teacher education students at California State University, Sacramento, Sacramento, CA 95819. He teaches mathematics education courses and supervises field work.

Support from Research

Much research into manipulative materials and their benefits for children has been conducted over the years. Although no single study validates the claim that children should use manipulative materials as they learn mathematics, the collective message garnered from many studies is that the materials are worthwhile.

Fennema (1972) summarized research on Cuisenaire rods versus a traditional approach. The research favored the concrete materials in first grade for the total range of work done in that grade. The value of the rods for second and third graders was less conclusive. She concluded that "there is some indication that children learn better when the learning environment includes a predominance of experiences with models suited to the children's level of cognitive development" (Fennema 1972, 637). She also summarized studies about other materials and stated that "these data tend to support the hypothesis that a learning environment embodying representational models suited to the developmental level of the learner facilitates learning better than a learning environment that ignores the developmental level of the learner" (Fennema 1972, 637). Fennema supports the use of many materials at the early levels of learning, with a gradual decrease in their use as children are able to handle concepts more symbolically.

Research supports the use of manipulatives at all school levels.

Manipulative materials are more widely used in the primary grades than in the upper grades of elementary school. However, research supports their use at all school levels. According to Driscoll (1981), manipulatives not only help children in the intermediate grades develop new concepts but can also be used to provide remedial help. "If there is any risk related to the use of manipulatives in these grades, it derives from their being ignored or abandoned too quickly" (Driscoll 1981, 24).

Suydam (1984) says that lessons that incorporate manipulative materials effectively have a higher probability of producing greater learning than lessons not incorporating such materials. She states that achievement is enhanced across a variety of topics and at every grade, achievement, and ability level (Suydam 1984, 27). The mathematical concepts presented to children in the middle grades are just as abstract and difficult to understand as those presented to younger children. Advanced work with whole numbers, common and decimal fractions and operations on them, measurement concepts, geometry, and many other topics that middle graders deal with are all better understood when children use manipulative materials in appropriate ways.

Support from the Classroom

Each issue of the *Arithmetic Teacher* contains several articles that feature uses of manipulative materials. Most of the articles are prepared by mathematics educators who are convinced of the value of manipulative materials for children. Some are prepared by teachers who know the benefits of a particular material for their students and who write to share their enthusiasm with others.

A middle school teacher, Herbert (1985, 4) wrote that manipulatives allow teachers to create situations that draw mathematical responses from children. Such situations result "in improvements in motivation, involvement, understanding, and achievement—overwhelming reasons to believe that manipulatives are good mathematics."

Sullivan (1981) describes a series of activities that involve play money. She concludes that the materials can be applied to work with percentages, decimal and common fractions, and number bases to help children better understand these areas of mathematics. She also cites the motivational value of money used as a learning aid.

Three sets of materials for teaching rational numbers in the middle grades are described by Jacobson (1984). The models can easily be constructed by children themselves and offer a basis for understanding decimal fractions.

The *Mathematics Their Way* program developed by Baratta-Lorton (1976) places heavy emphasis on manipulative materials. This program has enjoyed success in many classrooms across the country. In the *Mathematics the Piaget Way* program (Hendrickson 1981) used in the Tower-Soudan Schools in Minnesota, manipulative materials also serve as the basis for concept development in mathematics.

The experiences of individual teachers and school systems who have developed programs that regularly involve manipulative materials demonstrate conclusively that such programs can be used at all school levels. Equally important is the demonstration of the fact that manipulative materials do enhance the learning of mathematics.

Concluding Statement

Learning theories and evidence from research and classroom practice support the use of manipulative materials to help children learn and understand mathematics. Well-chosen and properly used manipulative materials enhance children's learning, generate interest, relieve boredom, and promote problem-solving and computational skills.

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Research Report

Manipulative Materials

In response to questionnaires, most teachers indicate that they believe that manipulative materials (chips, blocks, fraction pieces, etc.) should be used for mathematics instruction. Children should be involved in the process of doing mathematics, and the use of concrete materials is integrally related to the development of meaning. As children work with objects and talk about what they're doing, they begin to see relationships to learn mathematics.

Yet belief is not always translated into action. First-grade teachers report rather frequent use of manipulative materials. But teachers from grade 2 on indicate less and less use of materials.

As we are faced with widespread concern about achievement, it seems important to consider the evidence from research:

- Lessons using manipulative materials have a higher probability of producing greater mathematics achievement than do lessons in which such materials are not used. This finding presumes that using manipulatives is plausible in a lesson - they can't be used with all topics or for all purposes.

- Achievement is enhanced across a variety of topics, at every grade level K-8, at every achievement level, at every ability level.

- Children need not necessarily manipulate materials themselves for all lessons, however. Watching the teacher use the materials in a demonstration is sometimes at least as effective. This result may be because directing children's attention to important mathematical ideas is easier when the teacher is in control of the materials.

Only one caution is needed: not all children need to use manipulatives for the same amount of time. Prolonged use may keep some children using procedures too simple and inefficient for them. Concern for individual needs must govern the use of manipulative materials.

** Suydam, Marilyn N., and Jon L. Higgins. Activity-based Learning in Elementary School Mathematics: Recommendations from Research Columbus, Ohio: Eric/Smeac, 1977.

THE PURPOSE OF MANIPULATIVES

* to build basic concepts

* to provide a means for students to think for themselves

* to represent persons or objects related to an actual experience

* to provide a concrete approach to the learning of mathematics

* to help students learn through a multi-sensory approach by actually feeling and seeing what they are doing

* to help students gain insight into the "what", "why" and "how" of an experience

* to prepare students for higher levels of representation with numerals and symbols

* to prepare students for more sophisticated levels of thinking

* to support and encourage student verbalization during learning experience

* to check for understanding rather than just memorization

* the check for accuracy when used diagnostically

* to allow students to physically represent their mental imagery and share with other students

* to help students gain a feeling of confidence

* to discourage parrot-like responses from students

* to stimulate, enrich and vitalize teaching by developing concepts through the use of alternative models and strategies

Manipulatives—What, Why, When, and How

Although the language of mathematics is almost entirely symbolic, the most powerful tools in teaching mathematics are concrete. Yet surprisingly few teachers take advantage of these materials, often called "manipulatives."

WHAT are manipulatives? Manipulatives are learning materials that appeal to several senses and involve students in the active exploration of math principles and relationships. Some of the most popular manipulatives are everyday objects like buttons, beads, macaroni, beans, containers of various sizes and shapes—materials that are easily collected by teachers themselves or by contributions from students and parents. Because of their inherent familiarity, these materials are especially effective in demystifying math and in providing a firm basis for mathematical thought in the real world.

However, not all mathematical abstractions are easily modeled with found objects. Math teachers over the years have needed to create specially designed materials for certain math concepts. Examples are attribute pieces, base-ten blocks, and Unifix cubes. Although versions of these materials may be teacher-made, commercially produced materials often offer certain advantages. One advantage is durability. Materials made of plastic or wood generally outlast those made of cardboard or paper. Another advantage of commercially produced materials is convenience. Not all manipulatives are easy to make. The time it takes to mass produce items like bean sticks, for example, may not be justified by the savings.

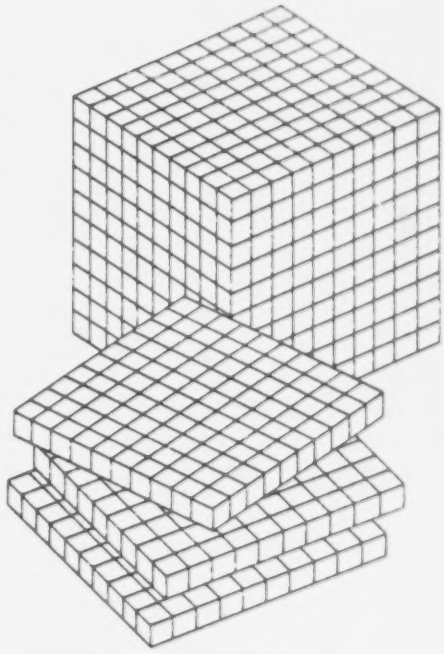
A final advantage of commercially produced materials—of particular significance for young children who are easily distracted—is that they are usually designed so as to minimize the amount of interference from nonessential elements, such as the number of holes in a button. Instead, these materials (Unifix cubes, for example) often rely heavily on factors like color to enhance meaning.

Whether teacher-made or commercially produced, the best manipulatives offer a variety of visual, auditory, and tactile clues that can be tapped to bring meaning to mathematical ideas. Their bright colors, interesting textures, and satisfying shapes are guaranteed to invite exploration.

WHY use manipulatives? Three of the most prominent theorists on the development of mathematical thinking—Jerome Bruner, Jean Piaget, and Zoltan P. Dienes—all agree that in order to understand abstract concepts, children must first manipulate concrete representations of the concepts. If the goal is to learn the addition algorithm, for example, students must first work with concrete objects like base-ten blocks before they can understand the whys and hows of trading. Once the concepts have been grasped with concrete materials, students can begin to transfer their understanding to graphic representations (pictures of base-ten blocks, for example) and finally to abstract symbols. The learning builds from the concrete through increasingly more graphic and abstract modes in three distinct stages:

The Road to Abstraction

1. ***Concrete***
2. ***Graphic***
3. ***Abstract***



By the time students have arrived at the abstract mode, their understanding is based on a solid foundation of meaning rather than on rules that are easily forgotten.

The premise that the manipulation of concrete objects is necessary before abstract understanding can be attained has been the topic of many research studies. In general, the findings provide compelling evidence that work with manipulatives—especially at the primary level—can lead to significantly greater understanding of mathematical concepts—even to higher scores on achievement tests.

Support for manipulative-based instruction is found in all areas of the mathematics curriculum—including problem solving. In one study, kindergarten and first grade children who were given concrete materials to manipulate, showed a surprising ability to solve problems for which they had no previous formal instruction.

When should manipulatives be used?
Though many studies forcefully suggest that primary school children are being exposed to symbols and abstractions far too early, no one espouses the idea that mathematics instruction should consist exclusively of the use of manipulatives. Ultimately, the goal must be a steady progression toward mathematical abstraction. The optimal time for advancing beyond manipulatives to more abstract modes depends on the developmental level of the students.

Children under the age of 7 or 8, described "preoperational" by Piaget, are perception bound. Because they are totally dependent on the concrete manipulation of objects for understanding, their ability to manipulate symbols is extremely limited. Also, their perceptions are easily influenced by irrelevant factors. By second or third grade, however, children enter the more advanced stage of "concrete operations." During this phase, they are still dependent on perception for understanding but they can organize abstract ideas in an increasingly sophisticated way. It is not until adolescence that children attain "formal operations"; only at this stage are they capable of deducing abstract ideas from other abstract ideas.

Although most studies of manipulative-based instruction have focused on the preoperational and concrete operational child in the math class, other studies have shown clearly that all children—and adults as well—benefit significantly when new concepts are developed through the manipulation of concrete materials.

HOW should manipulatives be used?
In order for manipulatives to be effective, teachers must take an active role in directing student discoveries and guiding interaction with the materials. Equally important, they must carefully groom students to make the all-important connection between the concrete representation of a concept and its graphic and symbolic forms. Detailed suggestions for using manipulatives to help students understand specific concepts can be found in many sections of the Teacher's Edition of *Addison-Wesley Mathematics*. Ideas for Getting Started, Activities That Count, Homework Bonus, Geometry Activities, and Ideas That Work are all rich sources of ideas for using manipulatives effectively.

In addition, student lessons are often specifically designed to provide a bridge from the hands-on activities in the Ideas for Getting Started to the symbolic form in the practice exercises. A lesson on two-digit addition, for example, might start out with a prebook activity in which students use base-ten blocks to model a problem (concrete mode). Students then work through a similar problem on the textbook page using illustrations of base-ten blocks (graphic mode). Each step in the process is then demonstrated in its corresponding symbolic form (abstract mode). When students are guided through an algorithm this way, they are better able to make a smooth transition from the concrete to the abstract.

As students grow in cognitive skills, the kind of manipulative chosen to model a mathematical structure can be increasingly sophisticated. For place value concepts, for example, a logical progression is the following:

Bundling sticks: Tens and hundreds bundles can be physically separated into individual sticks and counted.

Base-ten blocks: One ten looks like 10 ones glued together; one hundred looks like 10 tens glued together; and so on.

Chip trading: Hundreds, tens, and ones are identical except for color and placement.

Unicolor abacus: Hundreds, tens, and ones are identical except for placement.

Not all students respond equally well to each kind of manipulative. Some children exhibit a strong preference for discontinuous objects such as counters. Others favor continuous objects such as base-ten blocks. By presenting different concrete representations of the same mathematical principle, teachers provide for these individual learning styles. Multiple models also strengthen the generalizations that are derived. For these reasons, Dienes recommends that children "encounter the same mathematical structure dressed up in a number of different ways."

For young children, directed activities with manipulative materials are most successful when they are preceded by a period of free exploration. Ample time should be allowed for touching, observing, sorting, and playing with the objects. Once the students' natural curiosity is satisfied, they are ready to move on to more structured learning.

Do children become dependent on manipulatives for solving problems and working through algorithms? Indications are that the answer to this question is *no*. Since manipulatives are inherently cumbersome, students naturally graduate to more abstract models for representation. In the meantime, however, teachers do well to provide easy access to manipulative supports so that students of all levels have the opportunity to revert to their use whenever symbolic or graphic representations are inadequate for understanding.

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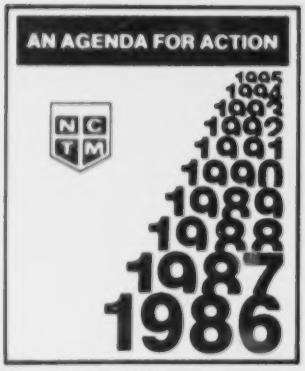
One Point of View

Using Manipulatives

By George W. Bright



Manipulatives hold the promise for helping many students understand mathematics, but as with any "cure" they also hold the potential for harm if they are used badly. Careless use of manipulatives may convince students that two mathematical worlds exist, symbols and manipulatives, and that it is acceptable for these worlds to have different rules. An acceptance of this inconsistency is certainly harmful to the development of proper mathematical understanding.

Students may appear to misunderstand concepts when they get different answers with symbols than they do with manipulatives. This discrepancy may, however, be an artifact of the physical constraints of manipulatives. Frequently, manipulatives almost force students to produce the correct answer, so laying the blame on the symbolic world may be inappropriate. Rather, it should be recognized that proper mathematical understanding simply does not exist when students do not see the need for accurate connection between the two worlds.

Indeed, the main emphasis that needs to be made in teaching with manipulatives is that symbols and manipulatives must always reflect the same concepts. Students don't automatically grasp this connection.

The acceptance of this need for consistency seems to have been an important step in my own appreciation for, and use of, mathematics. I suspect the same is true for many people. Many teachers, however, only *implicitly* embed this need for consistency between the two worlds in their teaching.

Oral language may provide one bridge between the two worlds. I believe that many mathematical concepts are developed through abstraction and generalization across multiple instances. However, abstraction need not always be done in a written symbolic mode. It is equally important to abstract in an oral language mode. Manipulatives can provide the experiential base for oral language while simultaneously complementing the symbolic side of mathematics. The experience of touching and moving objects may also serve to relate concepts sequentially, thus implicitly illustrating relationships among concepts. Unless the connection between the two settings is carefully articulated, however, counterproductive tension may be created between them.

Students' misunderstanding of the connection, as with any other misunderstanding, requires that remediation be undertaken. Unfortunately, little seems to be known about how to provide this remediation. But we must not let our lack of knowledge become a justification for substituting comfortable symbol-pushing techniques for instruction with manipulatives. Rather, we should endeavor to become more sensitive to finding adequate ways of incorporating manipulatives into instruction and helping students make appropriate connections between manipulatives and symbols.

Certainly manipulatives should not be used for teaching every concept. Careful use of manipulatives provides opportunities for generalizing across multiple instances of concepts. But as with any attempt to foster transfer of understanding, these general learning skills must be explicitly highlighted in instruction if they are to be learned. Learning how to connect the various parts of mathematics, thereby developing adequate generalization skills, is far more important than learning the particular concepts and skills presented in lessons built around manipulatives. All of us who teach mathematics must learn how to teach these general skills, and one of the tools for doing so is the use of manipulatives. ■

George Bright teaches computer education courses at the University of Houston, Houston, TX 77004, though he remains active in research on ways of improving mathematics instruction.

Readers are encouraged to react to these editorials by writing to the author, with copies to the Arithmetic Teacher *for consideration in "Readers' Dialogue." Please double-space all letters that are to be considered for publication.*

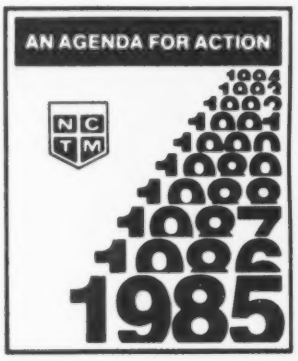
This Month's *MT*

Readers of the *Arithmetic Teacher* would enjoy the following articles in this month's *Mathematics Teacher:*

- "The Impact of the Black Death (1348–1405) on World Population: Then and Now," Michael G. Gonzales and William J. Carr
- "Activities: Calculators, Quotients, and Number Patterns," David R. Duncan and Bonnie H. Litwiller
- "Sharing Teaching Ideas: Graphs, Patterns, Periodic Functions, and Problem Solving," Ernest Woodward

One Point of View

Manipulatives Are Good Mathematics!



By **Elizabeth Herbert**
Houston Independent School District, Houston, TX 77027

When asked, "Do you like mathematics? Why or why not?" at the beginning of each school year, a startling 80 to 85 percent of my middle school students would reply that they did not like mathematics because it was boring and because they did not understand it very well. Their negative attitudes appeared to stem from a lack of understanding based on plain boredom with uninteresting classes.

For fifteen years, this informal survey yielded the same answers on the first or second day of school. By week three, the same students no longer dreaded mathematics. To their surprise and my delight, they were enjoying mathematics. Why? Manipulatives.

Manipulatives can and should be used on a regular basis in the mathematics classroom, along with a variety of other tools and methods. On the basis of my experience as a teacher, mathematics department head, instructional supervisor, and workshop leader, I strongly advocate the use of manipulatives for three reasons: manipulatives motivate students; manipulatives stimulate students to think mathematically; and manipulatives informally introduce "big" ideas in mathematics.

Teachers often claim that not enough time is available to use manipulatives, that their use is the same as playing games, and that they are difficult to manage with large numbers of students.

Yes, using manipulatives takes time. Isn't it better to take time at the beginning of a unit to motivate students and spark their thinking than to take time during a unit to bore them—and yourself—with endless drills and worksheets? Thinking time is time well spent.

Yes, using manipulatives feels like playing games. Teachers and students having fun thinking and discovering together is what makes teaching so rewarding. We must be ready to share ourselves, our thought processes, our own mistakes, and our sense of fun with students.

No, manipulatives are not difficult to manage with large numbers of students. First, explain to the entire class the purpose of the manipulative. Show them how it works and then pose a problem or situation. Divide the class into small groups and supervise them as you would any group activity. Circulate among them, help any who really need assistance, but try to let them struggle and experiment. Don't solve problems for them. If any students start flipping rubber bands or throwing rods, isolate them and make them work out of the textbook.

Conquer the fear of losing control of the class. A little noise can be expected when students work together. A few "ahs," "wows," and "I understands" will be music to your ears. Invite the principal to observe. If you're lucky, the principal will join a group and try to help solve a problem.

A favorite definition of mathematics states, "Mathematics is discovering relationships and expressing those relationships in symbolic form." What better way to introduce young children to mathematical thinking than to have them use Cuisenaire rods or fraction bars to discover the algorithms for adding and subtracting fractions? Middle school students can use geoboards to discover and write the formulas for finding the areas of polygons. Algebra students can feel challenged and interested when they are confronted with this proposal: "Let's find a way to determine the distance between these two points on your geoboard." The students use the Pythagorean theorem and then discover, write, and use the distance formula.

In each example, the manipulative provides the situation from which real mathematical thinking arises. The students gain the understanding that may have come using traditional methods after hours of drill and practice. Effective thinking in mathematics depends on understanding, not on remembering words and phrases from a text.

A confining, narrow textbook approach to teaching mathematics is inadequate and inappropriate in today's schools. It does motivate students—to take as few mathematics courses as possible—and often does not help them to become independent, confident problem solvers.

Textbooks transmit known mathematics; they tend to simplify learning by emphasizing the mastery of small, isolated steps. Mathematical thinking is not done in small, isolated steps using finished definitions from a text. Rather, it starts with situations that draw mathematical responses from the students. Manipulatives allow teachers to create those situations and involve the students actively in the creation of mathematics, resulting in improvements in motivation, involvement, understanding, and achievement—overwhelming reasons to believe that manipulatives are good mathematics. ●

MATHEMATICS MANIPULATIVES

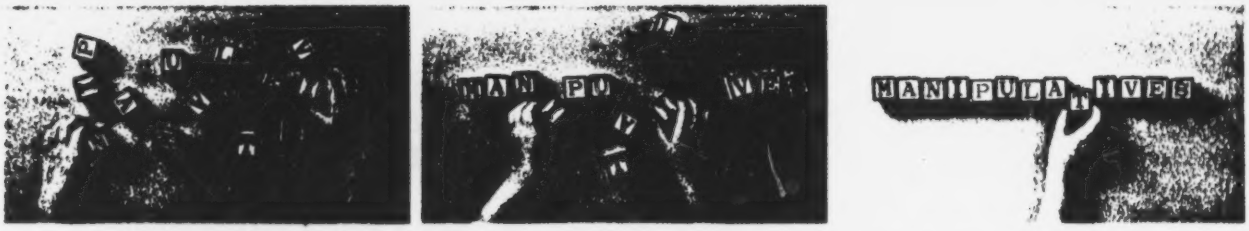
Manipulatives and Learning Stages

Concrete - Use physical materials with verbal directions. The student manipulates materials and responds orally.

Semi-Concrete - Use pictorial clues with verbal directions. The student manipulates materials and responds by showing pictures of the results.

Semi-Abstract (Symbolic) - Use pictorial clues and coded symbols. This is the "connection" stage. Students do not use manipulative materials.

Abstract - Give directions/problems with coded symbols. Students respond with coded symbols. Students do not manipulate physical materials.



Bridging the Gap between the Concrete and the Abstract

By **James W. Heddens**

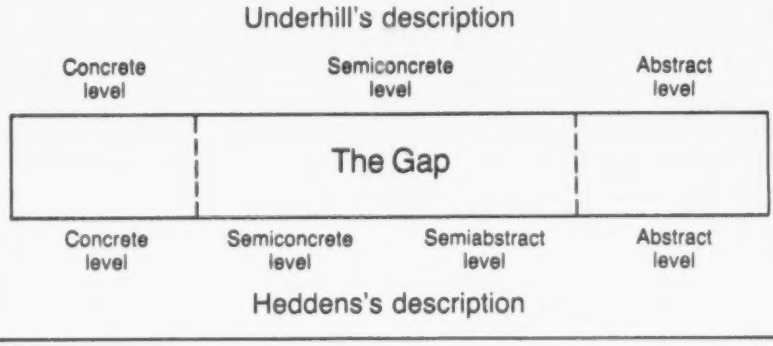
How can teachers help students make the transition from the use of manipulative materials to abstract mathematical symbols? Many students have difficulty understanding mathematics because they are unable to make a connection between the physical world and the abstract world.

Defining the gap

Underhill (1972) defines the stage between the concrete and abstract levels as the semiconcrete stage. Heddens (1984) divides this stage into two levels—semiconcrete and semiabstract. The semiconcrete level is a representation of a real situation; pictures of the real items are used rather than the items themselves. The semiabstract level involves a symbolic representation of concrete items, but the symbols or pictures do not look like the objects for which they stand. Tallies might be used to represent the idea of automobiles, for example.

The gap between concrete and abstract functioning should be considered as a continuum. Helping children bridge this gap is crucial because many children cannot cross it without the teacher's assistance. Figure 1 represents a continuum of learning, with the gap appearing between the concrete and abstract levels. Learners must internalize new knowledge at the concrete level and systematically progress along the continuum to arrive at the abstract representation of that knowledge.

Fig. 1 A continuum of learning from concrete to abstract



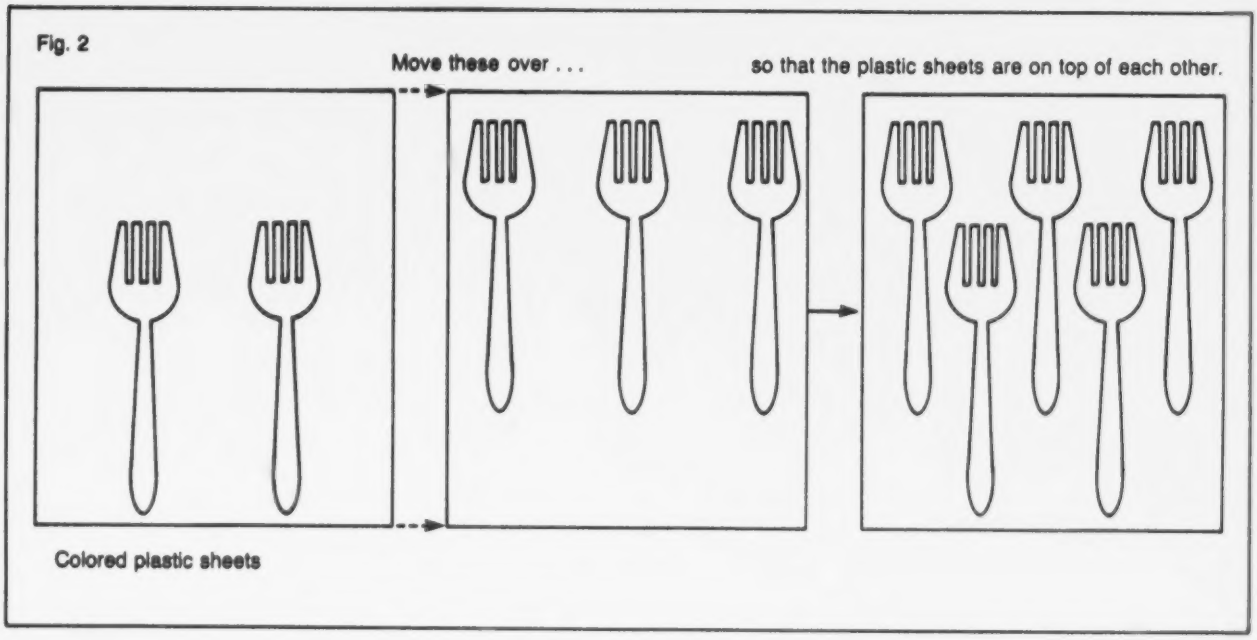
Two processes of interaction between reality and the mind are accommodation and assimilation. Piaget refers to these processes as *equilibration*. New knowledge can be assimilated very rapidly by some children. Other children, who must accommodate, or reorganize, their mental structures to incorporate new knowledge, need considerably more time. It takes time to think! While students who must accommodate are processing new material, teachers very often continue to present more new material, thus leaving the student in the lurch. A provision must be made for students to bridge the gap at different rates.

Bridging the gap with questions

Teachers can also help students bridge the gap by very carefully asking questions that will help children move from the concrete to the abstract level. Too many questions in mathematics classes require specific answers and encourage students to remember only facts. The students' thought processes are often ignored when questions are asked. Teachers should continually evaluate the quality and level of the questions that students are asked. Are the questions intellectually stimulating? Do they require students to synthesize before they respond? Questions that match all levels of Bloom's taxonomy (Bloom et al. 1977) should be used. More emphasis should be placed on the "how" and "why" questions and less emphasis on the "what" questions. Through systematic questioning by the teacher, children can begin to develop their own thought processes and use these techniques in their own thinking. Simply using manipulatives in teaching mathematics is not sufficient; teachers must guide children to develop skills in thinking.

Children do not learn mathematics

James Heddens is professor emeritus at Kent State University, Kent, OH 44242. He is interested in training teachers in the techniques of teaching mathematics to children. He has taught mathematics methods and educational computing to preservice and in-service teachers.



by being told about mathematics; children must actively participate in thinking mathematically. Teachers need to ask crucial questions that guide children to think through the mathematical concepts being studied. Questions asked by teachers can reveal new directions of thought, encourage children to continue their current line of thought, or provide clues that will stimulate thinking when progress has temporarily been blocked.

Children also need to be given opportunities to verbalize their thought processes; verbal interaction with peers will help learners clarify their own thinking. The development and verbalization of thought processes is time-consuming; the critical variable, however, is not time—it is thinking.

Bridging the gap with activities

Activities involving pictures of objects, textbook illustrations, models on the overhead projector, and drawing, as well as demonstrations by teachers and peers, can smooth the transition between concrete and abstract functioning.

Mathematical concepts emerge from perceptually unrelated ideas that are generated as the learner participates in concrete experiences. The bridging of the gap is the result of a process of internalization. The learner consciously builds on many concrete experiences and emerges with mathematical concepts at the abstract, symbolic level. Teachers must be the catalysts who assist students with this process.

Thought-processing skills, such as the use of logical thinking to understand similarities and differences, can help children bridge the gap from the concrete to the abstract. Thought-processing skills can be developed with such materials as attribute blocks or people pieces. Primary level mathematics students should have many experiences in observing and verbalizing the characteristics of the selected materials. For example, children should be asked to compare two attribute blocks. Then all the attribute blocks can be placed in front of the children, who then sort them so that all the blocks in one pile are alike in some way and all the blocks in the second pile are alike in another way. Children should be encouraged to think of other ways of sorting the blocks. For example, blocks with corners could be placed in one pile and blocks with no corners could be placed in a second pile; or, all the red blocks could be placed in one pile and all the blocks that are not red in a second pile. The same principle can be used to separate the blocks into three piles, then four piles, and so on. These games can enhance students' thinking skills. Looking for patterns and extending them in this manner are excellent ways of developing thinking techniques. An examination of patterns should also be done using number sequences and the place-value system.

A specific example of bridging the gap

Teaching place value is often done at an abstract level, with children using only printed symbols. An abacus with beads of different colors representing different place values is a way of presenting place-value concepts to children on the semiabstract level. The abacus is almost as abstract as place value itself. To bridge the gap we must start on the concrete level and move to the abstract level.

The teaching of place value at the concrete level should begin with blocks or with the actual bundling of ten counters into a package to be moved from the ones place to the tens place. Packages of gum with ten sticks in a package are excellent models to use in teaching grouping by tens. Pictures of blocks or of bundles of ten counters would be appropriate to

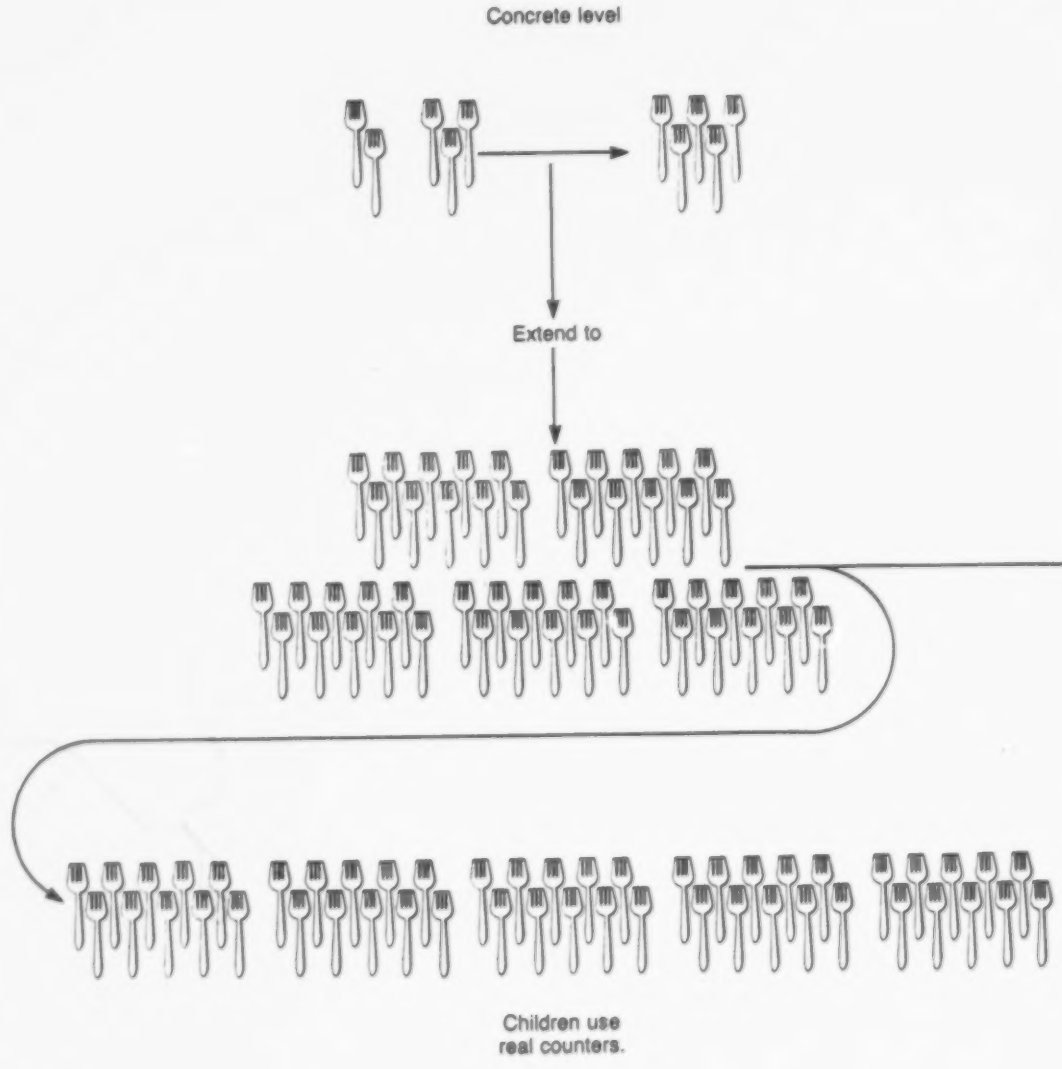
move to the semiconcrete level. The next level, semiabstract, could involve the use of tallies in a place-value chart. All these activities would lead to the abstract level—using symbols to represent place value.

The first step in teaching the basic operations with whole numbers occurs at the concrete level. Children must manipulate things to develop a model for each operation. For example, sheets of paper with plastic forks on them can represent discrete sets; then the two sets can be moved together to obtain a sum for the addition model (see fig. 2).

It is important that children not make a third set to represent the sum; they must place the two disjoint sets together to obtain one set that represents the sum. Demonstrations in which teachers or peers use a variety of models can aid in a student's movement along the continuum. Demonstrations with children would be at the concrete level. The semiconcrete level would involve using pictures of objects. The semiabstract level would involve tallies on the chalkboard or an overhead transparency. The abstract level is reached when the mathematical concept is recorded in a mathematical sentence, such as $2 + 3 = 5$.

Figure 3 illustrates how the gap can be bridged in teaching place value. The concept is extended from the ones place to the tens place. This figure shows how the place-value and addition concepts are brought togeth-

Fig. 3



er to develop meaning for addition of multiples of tens. This idea can now be extended to the addition of ones and tens.

Teachers need to orchestrate mathematical concept development very carefully to provide a smooth transition from the concrete to the abstract levels. The need for a careful sequencing of activities to lessen students' dependence on the concrete level and increase their facility with the abstract level is crucial.

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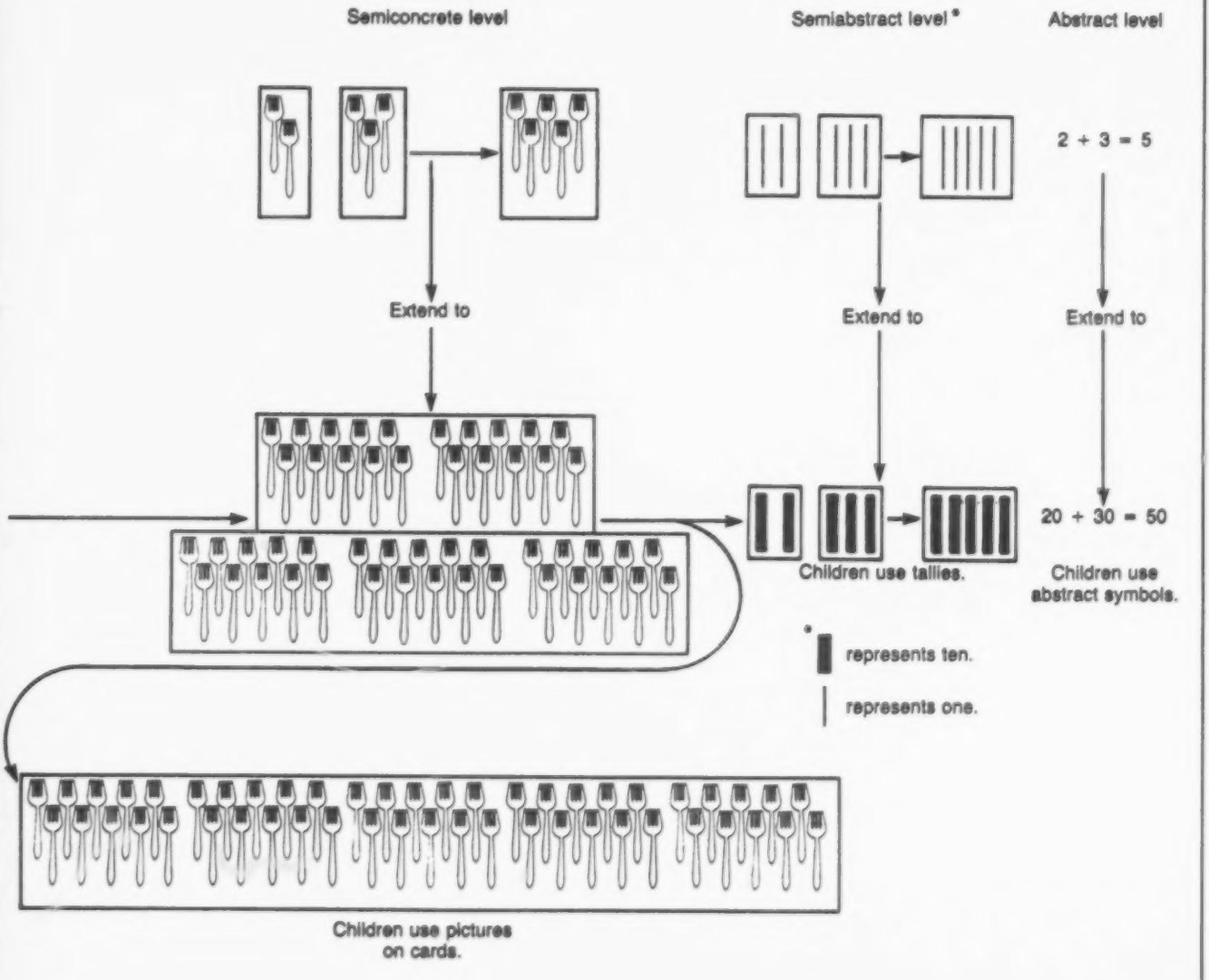

Modeling Operations and Algorithms

By **Ian D. Beattie**

The idea of using manipulative materials to model operations and algorithms is not new. The *Arithmetic Teacher* frequently includes articles on the topic, and its readers should have accumulated a wealth of excellent teaching suggestions. Current methodology books and school texts place great emphasis on the use of manipulatives. Thus one might expect appropriate manipulative materials to be a mainstay of every mathematics classroom, but evidence suggests otherwise. Although teachers *believe* that manipulative materials should be used for mathematics instruction and that the use of such materials *does* develop understanding, the use of manipulative materials diminishes through the grades (Suydam 1984a).

Several reasons for this situation might be put forward. First, each operation has several interpretations or meanings, but only one standard algorithm is taught. In the solution of word problems, once the appropriate operation is chosen, the algorithm is performed without reference to the interpretation of the operation embedded in the word problem. That is, the algorithm becomes a rote mechanical procedure. In fact, after the algorithm is understood, it *should* be done by rote. One should not have to think through every simple computation. This rote notion of an algorithm is reinforced by pages of computations that *have* no reference to any particular interpretation and by the fact that this method is how teachers themselves *learned* the algorithms. This situation leads to the second reason.

Preservice teachers and many experienced teachers have great difficulty making the connections among the meanings of an operation, the associated manipulation of materials, the verbal explanations that accompany the manipulations, and the steps of the algorithm.

Third, teachers are pragmatic. Achievement is measured by how well students *do* the algorithms, not how well they understand them, and children must have adequate mastery of facts and algorithms if they are to be successful in later grades. Consequently, teachers focus instruction on mastery of the mechanics of algorithms and not on understanding. With the increasing emphasis on mastering algorithms in the upper grades, the decrease in use of manipulatives is most likely inevitable. Probably inevitable, too, is the type of meaningless explanation that follows. It was given by a sixth-grade remedial student, and it is an extreme, but not uncommon, example of rote explanation:

> "9 doesn't go into 8. Borrow from 2, cross out 2, put 1 there and 1 there. 9 from 18 is 9. 3 doesn't go into 1, cross out 5, put 4, another 1 beside 1, equals 11. 3 take away 11 is 8, bring down 4."

Manipulative materials are the key to understanding operations and algorithms.

$$\begin{array}{r} {}^{4}\not{5}\ {}^{11}\not{2}\ {}^{1}8 \\ -\quad 3\ \ 9 \\ \hline 4\ \ 8\ \ 9 \end{array}$$

Those who work in the area of diagnosis and remediation come across many such examples of rote procedures and inappropriate language. The use of such terms as *borrow*, *carry*, *bring down*, and *cross out* is common across all ability levels.

A great deal of time is spent on teaching the standard algorithms for whole and rational numbers. Usiskin (1983) is probably correct when he writes, "A century from now people will gasp when they learn that six to nine years were spent getting good at pencil and paper arithmetic. They will ask: Didn't they have calculators then?" Does this mean that the teaching of algorithms should be discontinued? No, but it does mean that the *reason* for teaching algorithms should change from obtaining correct answers through the rote manipulation of symbols to understanding the meaning of the operation and the rationale behind each step in the algorithm. And the *way* in which algorithms are taught should change from a mechanical procedure to methods that will facilitate *understanding*. This understanding will become more, not less, important in the calculator and computer age.

Few elementary school teachers would wish to explain the addition algorithm as did Dodge (1969):

> To add 23 and 48 we write

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$$
\begin{aligned}
23 + 48 &= (2 \cdot 10 + 3) + (4 \cdot 10 + 8) \\
&= (2 \cdot 10 + 4 \cdot 10) + (3 + 8) \\
&= (2 + 4)10 + 3 + 8 \\
&= 6 \cdot 10 + 11 \\
&= 6 \cdot 10 + (1 \cdot 10 + 1) \\
&= (6 \cdot 10 + 1 \cdot 10) + 1 \\
&= (6 + 1)10 + 1 \\
&= 7(10) + 1 \\
&= 71
\end{aligned}
$$

. . . We have obtained 23 + 48 = 71 in a most tedious manner. How much easier to write:

$$
\begin{array}{r}
23 \\
+\ 48 \\
\hline
71
\end{array}
$$

But why can it be used? What justifies it? . . . The long form for addition explains the short algorithm. It demonstrated why the short form works and how its carrying is justified.

Tedious indeed! But Dodge's explanation does stress that an algorithm is more than a rote procedure. It is a systematic, iterative application of the basic properties of our numeration system.

Evidence shows that teaching for understanding of operations and algorithms will help computational mastery. In reviewing research on fractions, Suydam (1984b) concluded that "we need to shift the emphasis from having students learn rules for operations on fractions to helping them develop conceptual bases for fractions." Again dealing with fractions, Driscoll (1984) observed that "once the procedures for computing fractions are forgotten, few teenagers can reconstruct them," that "to use rational numbers a child must understand rational numbers," and that "all levels of students need experiences with concrete materials." The dramatic improvement of computational scores obtained by Wheatley (1984) in a study that emphasized problem solving and manipulative materials and deemphasized computation offers further support for the use of manipulative materials. Thus it appears that whether a teacher's goal is computational mastery or understanding, manipulative materials are important.

One might go further and argue that manipulative materials are the key to understanding operations and algorithms. In an excellent article Young (1983, 12) defined manipulative materials as "objects which represent mathematical ideas that can be abstracted through physical involvement with the objects." It is important to note that mathematical ideas are abstract, but the nature of human growth and development demands pedagogical approaches that involve representations of these ideas. Rousseau (1985) discussed the role of four kinds of representations: (1) concrete representations, which are three-dimensional and manipulable, such as base-ten blocks; (2) pictorial representations, such as number lines, graphs, and pictures of blocks; (3) language representations, which are the accepted oral and written vocabulary that has been given specialized meaning; and (4) alphanumeric representations, such as 2, +, 7, and the standard algorithms. When the teaching emphasis is placed on the fourth class of representations, algorithms are taught as rote procedures, language is inappropriately used, and, most important, understanding is lacking. The teaching emphasis should be on the connections among the meanings of an operation, the associated manipulation of materials, the verbal representations that accompany the manipulations, and the steps of the algorithm. Manipulative materials are the keys to making those connections in four important, distinct, but closely related, ways.

An understanding of algorithms will become more important in the calculator and computer age.

1. *Materials clarify the concept or meaning of each operation.* For each operation several different real-life situations are solved by application of that algorithm. Addition is applied to joining sets or to counting on; subtraction is appropriate for take away, comparison, missing addends, or partitioning of a set into two subsets; multiplication is appropriate for joining equivalent disjoint sets, for row-by-column, area, or Cartesian product situations; and division is applied when sharing out (partitive) or repeated subtraction (measurement) is called for. However, although the same algorithm is applied in each instance, the physical situations are different, and manipulative materials are essential for showing these differences.

2. *Materials clarify the language of each operation and algorithm.* As illustrated by the preceding subtraction example, a great deal of "gibberish" is associated with the algorithms. The description given has no relationship either to place-value concepts or to the physical manipulations that would be appropriate in arriving at a solution. Words like *borrow* and *carry* cannot be justified as other than rote verbal memory aids, and such words promote errors when students attempt to apply them to nondecimal situations, as in the subtraction of fractions when regrouping is necessary. The use of "minussing" and "timesing" for subtraction and multiplication, respectively, and "goes into" for division can hardly lead to understanding. Verbal descriptions *are* necessary, but they can and should be mathematically appropriate.

3. *Materials clarify the algorithm.* As mentioned earlier, algorithms are more than rote procedures: they are based on the positional notation and basic properties of our numeration system. Appropriate manipulative materials show these properties in action, and each step of an algorithm describes some action on the materials. An understanding of the algorithm is built by associating the steps of an algorithm with the actions on the manipulative materials.

4. *Materials clarify pictorial representations.* Textbooks rely heavily on the communication of ideas through pictures, but as Campbell (1981) has shown, the ability to interpret pictures is developmental—children have to learn how to interpret pictures. By modeling operations and algorithms with objects, children will be better prepared to interpret pictures that model operations and algorithms.

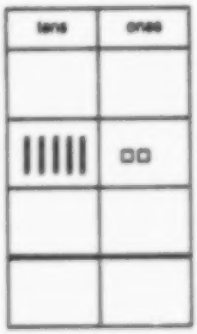
Different physical situations can model a given operation, but the standard algorithm is identical for each situation. For example, the several interpretations of the operation of

Fig. 1 Correlating manipulatives, language, and algorithm for two models of subtraction

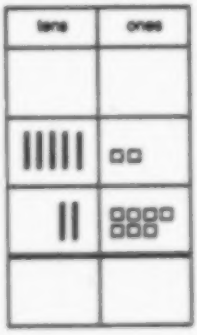
Problem A. "Jim had 52 marbles but lost 27. How many does he have left?"

Problem B. "Jim has 52 marbles. John has 27 marbles. How many more marbles does Jim have?"

1.



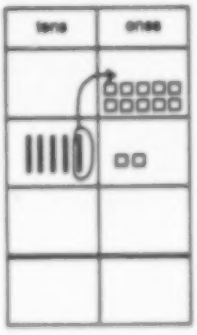
"Show 52, putting ones and tens in the correct columns."



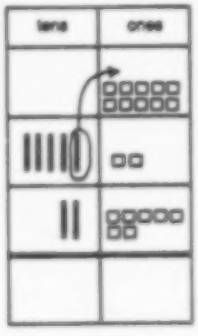
"Show 52 and 27, putting ones and tens in the correct columns."

$$\begin{array}{r} 52 \\ -27 \\ \hline \end{array}$$

2.



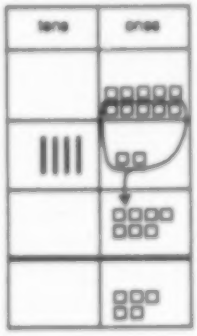
"You can't take 7 from 2. Regroup a ten into ten ones. This gives four tens and twelve ones."



"Compare the ones. 7 is greater than 2. Regroup a ten into ten ones. This gives four tens and twelve ones."

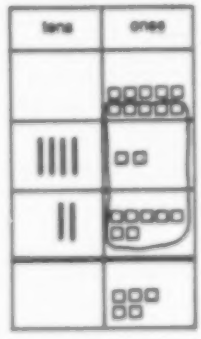
$$\begin{array}{r} {}^{4}\not{5}\,{}^{1}2 \\ -27 \\ \hline \end{array}$$

3.



"Take away seven ones."

"Show how many are left."

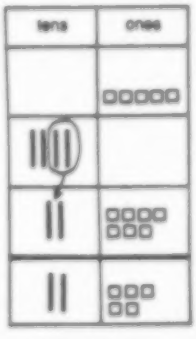


"Match up the ones."

"Show the difference."

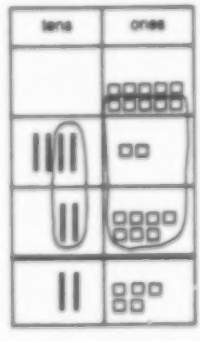
$$\begin{array}{r} {}^{4}\not{5}\,{}^{1}2 \\ -27 \\ \hline 5 \end{array}$$

4.



"Take away two tens."

"Show how many tens are left."



"Match up the tens."

"Show the difference."

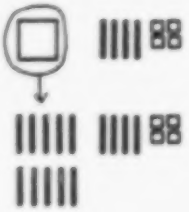
$$\begin{array}{r} {}^{4}\not{5}\,{}^{1}2 \\ -27 \\ \hline 25 \end{array}$$

Fig. 2 Correlating manipulatives, language, and algorithm for the partitive model of division

Susan has a package of 144 candy canes that she wants to share evenly among six friends. How many canes will each friend get?

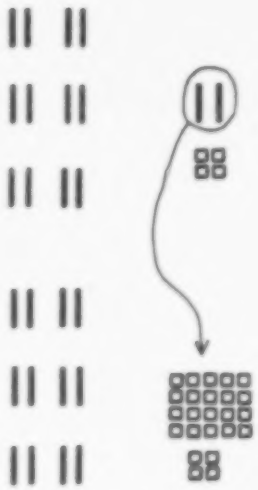
Show the 144 canes to be divided evenly among 6 friends.

$$6\overline{)144}$$



You can't give a hundred to each. One hundred is ten tens. So you have fourteen tens altogether.

$$\begin{array}{r|l} 6 & 14\,|\,4 \end{array}$$

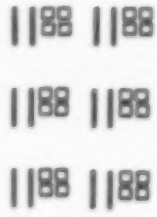


Give two tens to each friend and record 2 in the tens column. This partitioning takes twelve tens, leaves two tens.

$$\begin{array}{r|l} & 2 \\ 6 & 14\,|\,4 \\ & 12 \\ & 2 \end{array}$$

Each ten is ten ones. With the four left, this makes twenty-four ones.

$$\begin{array}{r|l} & 2 \\ 6 & 14\,|\,4 \\ & 12 \\ & 2\,|\,4 \end{array}$$



Give four canes to each friend and record a 4 in the ones column. This sharing takes twenty-four canes, and no canes are left.

$$\begin{array}{r|l} & 2\,|\,4 \\ 6 & 14\,|\,4 \\ & 12 \\ & 2\,|\,4 \\ & 2\,|\,4 \\ & 0 \end{array}$$

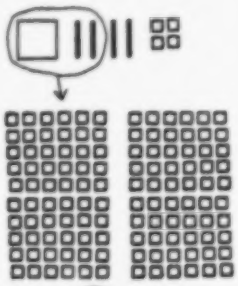
So each friend got 24 canes.

Fig. 3 Correlating manipulatives, language, and algorithm for measurement division

Susan has a package of 144 candy canes, enough for her to give 6 to each of her friends. How many friends can she share the candy canes with?

Show the 144 canes to be divided into groups of 6.

$$6\overline{)144}$$



You can't make a hundred groups of 6.

You *can* make twenty groups of 6. Twenty is two tens.

Record 2 in the tens column.

Take one hundred and two tens away.

Two tens are left.

$$\begin{array}{r} 2 \\ 6\overline{)144} \\ \underline{12} \\ 2 \end{array}$$

Each ten is ten ones. With the four left, this makes 24 ones.

$$\begin{array}{r} 2 \\ 6\overline{)144} \\ \underline{12} \\ 24 \end{array}$$

Make another four groups of 6. Record the 4 in the ones column. Take the 24 away.

$$\begin{array}{r} 24 \\ 6\overline{)144} \\ \underline{12} \\ 24 \\ \underline{24} \end{array}$$

There are 24 groups of 6 all together. So Susan can share with 24 friends.

subtraction are all solved by applying the standard subtraction algorithm. Manipulative materials model these situations, but actions on the materials, the verbal descriptions of the actions, and the steps of the algorithm must be coherent. What one *does*, what one *says*, and what one *writes* must match. Figure 1 shows how this matching can be done with two models of subtraction. The algorithms are identical, but the physical manipulations and associated explanations are different. The reader must, of course, infer the manipulations. The explanations can vary, provided they emphasize "taking away" for problem A and "comparing" for problem B.

Similarly, figures 2 and 3 show that for two models of subtraction, the algorithms are identical but the manipulations and explanations differ.

One final point can be made. A reduction in the amount of time spent on working with manipulative materials may have a short-term payoff in students' computational facility, but it minimizes their potential for transfer. It may be that many of the difficulties children encounter in such areas as fractions or decimals occur because children have learned rote procedures without fully understanding the operations or the algorithms for whole numbers.

Summary

Manipulative materials can play an important role in learning operations and algorithms. They can clarify the several interpretations of each operation, establish a basis for correct mathematical language, and show why algorithms work. Manipulatives can make sense out of nonsense.

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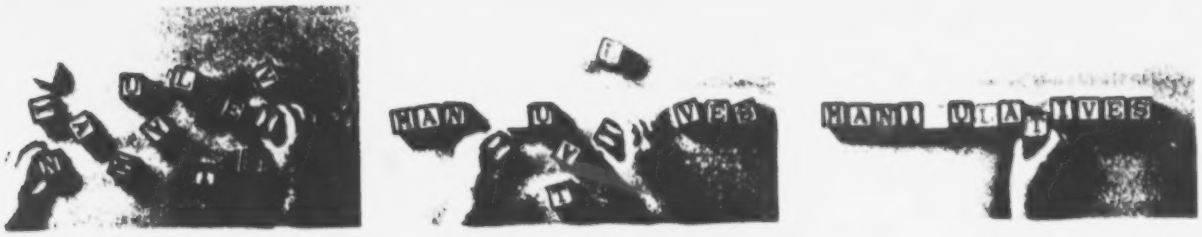

Special Learners

By **Carol A. Thornton** *and* **Barbara Wilmot**

Children with special learning needs fall into two distinctly different groups—the learning handicapped and the mathematically gifted. In this article consideration is given to the important role of manipulatives in mathematics instruction for each of these groups of students.

Using Manipulatives with the Learning Handicapped

Classroom teachers today work with larger proportions of handicapped students in their mathematics classes than ever before. Probably the biggest single obstacle for the learning handicapped is their inability to organize information to be learned. Developmental lags, poor retention, tendencies to be easily distracted, perceptual problems, logical-thinking or linguistic difficulties—any of these may account for the inability to organize. Each student is different, but most have one trait in common: they benefit from a carefully structured, active approach to learning mathematics.

As handicapped learners are guided in their work with materials and are given the chance to *verbalize what they do*, they begin to internalize and make sense of things in their own minds. Structured situations involving manipulatives help these students organize their thinking so that they begin to see relationships or follow the flow of a computational procedure. Four ideas for providing this structure follow.

1. *Vary the way you present manipulatives in a lesson to capitalize on students' learning strength.* When introducing or summarizing a manipulative activity, stagger the presentation of visual, verbal, and tactile information (Toohey 1984). For example, if a student is a strong visual learner, a silent visual demonstration might be a good introduction to a lesson. Or, for auditory learners, provide a verbal overview of what is to happen. At least initially, students do not have to process visual, auditory, and tactile information at the same time. This approach makes it possible for you to motivate students better and capture their attention at key points in a lesson.

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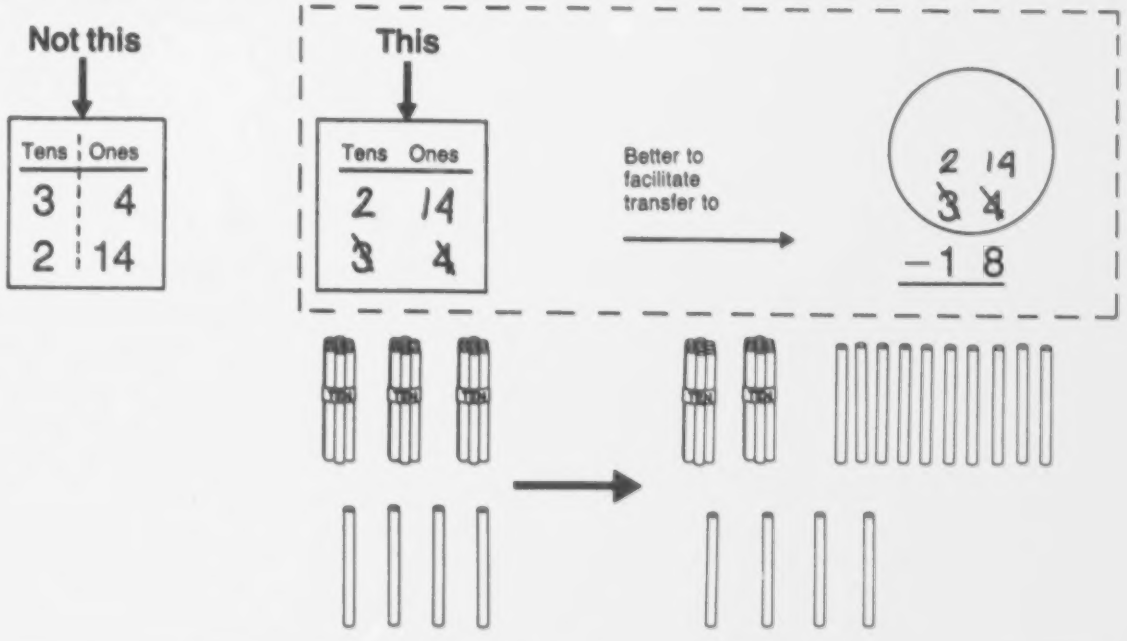
Fig. 1 This example of renaming illustrates an approach that is more effective for the learning handicapped student than the standard treatment offered in textbooks.



2. *Help students build important self-monitoring skills.* In addition to important concepts and skills, learning handicapped students must also be taught *how to learn* (Thornton and Toohey 1985). Central to this perspective is the development of self-monitoring skills, which are critical when students are under pressure (as in tests), are tired, or are under stress. Appropriate questions *repeated* during active instruction with materials, illustrated in the examples that follow (see idea 4), elicit responses during initial instruction. These same questions often are internalized and used later by the students themselves as the basis for selecting correct principles or procedures for completing mathematical tasks.

3. *Provide for consistency.* Often special educators and classroom teachers must cooperate in planning and implementing the mathematics curriculum for handicapped learners. It is critical that these teachers agree on ways to present content to students. Sometimes this may require refining textbook suggestions in prebook activities with manipulatives (see fig. 1).

4. *Carefully plan the use and the* phasing out *of manipulative materials both in concept and in skill learning.* The following six-step teaching sequence has proved helpful in work with most learning handicapped students.

1. Manipulate—talk

Use materials to model an example. Talk through each step. Unless children are highly distractible, involve them almost immediately in the manipulation. Those with attention disorders should watch and interact verbally during the first few examples so that, under the teacher's control, they can better attend to the thinking being highlighted.

2. Manipulate—write

Use manipulatives to model the example. Write up each step as it is completed. The goal now is to relate the manipulation of objects to what is written by having students *verbalize the thinking.* Learning handicaped students often are just told what to do. Since many of these students only internalize ideas when they *hear themselves speak,* they need to be led into describing procedures orally in order to learn them.

Example

$$\begin{array}{r} 34 \\ -18 \\ \hline \end{array}$$

Use bundles of tens and single sticks to model the example. "Four ones. Is that enough to take away 8?" (No, so invite students to trade using sticks, to describe verbally what was done, then to record to show now there are 2 tens and 14 ones.) Continue in a similar manner to subtract ones and then tens.

One must also structure the activity so that students do not become overly dependent on concrete representations. For example, do not allow children to count objects to find an answer to computational examples. If facts aren't mastered, make a fact list available until they can be worked on separately. Put the materials in a box

(Tucker 1983) or cover them with a hand after asking, "How many now?" Then, when an answer is given orally, students can count, *if necessary*, to check

3. Talk—write—manipulate: step by step

Talk through each step, write what you say, and use manipulatives after each step is recorded to check *(five minutes a day for several days)*. This step is the most critical for helping handicapped learners internalize appropriate language and thinking. Too often we stop short of this important step, which clinches initial mastery and fosters retention.

Example. Extend the work begun in step 2. Let students talk through each step (prompt if necessary). Students should now recognize that a trade is needed and should tell you that 3 tens and 4 ones become 2 tens and 14 ones after the trade. They would record this (as in the subtraction problem of fig. 1), then use the sticks to check that what they've written is correct. They then continue subtracting ones and tens in a similar manner.

4. Talk—write—manipulate: entire solution

Talk through the entire solution, recording as you go. Then model the entire example as a check. Given that attention to *students' talk*, to the *use of materials* to model procedures (step 2), and to checks of written answers (step 3) has been adequate, students typically are ready to carry on independently. At first require them to work each problem to the end, then to use materials to check their solutions. When children themselves can correctly demonstrate and explain the thinking they used to complete an example, then manipulatives are no longer needed.

5. Discriminate

Help students discriminate new from previous learning. For example, have students circle and do just those problems that require renaming when these are mixed with simpler problems with no renaming.

6. Compute—model

From time to time, use manipulatives to check written work. The goal now is maintenance. If this step is neglected, learning handicapped children will narrowly focus on getting answers, make conceptual errors, and fail in the transfer to related learning.

Similar instructional sequences have been developed and used with regular-class students by other mathematics educators (e.g., Tucker pers. com. 1981; Rathmell 1983). For the learning handicapped, however, such approaches are critical to success in mathematics and, therefore, are necessary for effective teaching of these special students. We suggest that verbal problem solving that relates written work to real-life applications be incorporated throughout. Otherwise, many handicapped learners will not make the transfer to using mathematics outside the classroom.

Using Manipulatives with the Mathematically Gifted

Manipulative materials are valuable tools for gifted students. When learning the basic facts and algorithms for the four operations with whole and rational numbers, these special students also need to have experiences at the concrete level. However, the number of examples necessary is usually considerably fewer than that for the other students because most gifted students have good visual imagery and can perceive relationships and make generalizations. For example, when students in a Saturday class of gifted fifth graders complained about long division, two examples (752 ÷ 3 and 409 ÷ 5) using partitive division with the base-ten blocks were adequate to allow them to understand what they were doing and how to record results.

In an enrichment curriculum for gifted students, concrete materials have an important place. Setting specific expectations for their use only limits potential outcomes. The learning of particular skills may be one purpose of an activity, but the overriding goal for gifted students should be to extend their thinking to higher levels, using the physical materials as a tool. The best approach is for the teacher to provide materials and general directions and then to ask appropriate questions (such as those that follow). Students will move in various directions according to their abilities, motivation, and previous experience.

The classroom climate should be informal so that students feel comfortable talking, moving about, and experimenting. Working in groups of two to four offers students an opportunity to interact with peers as they learn to brainstorm, analyze, cooperate, compromise, and evaluate their own abilities. Manipulative aids can be introduced in activities that are paced to meet each student's or group's needs. Telling should be minimized and discovering maximized. The teacher does not need to know the answers to all the questions and indeed should include questions that have no answer.

When used appropriately with gifted students, manipulative aids have at least six functions. They can—

1. build concepts and related vocabulary;
2. improve spatial visualization;
3. allow discovery of patterns and relationships;
4. provide problem-solving experiences;
5. teach the essence of verification or proof; and
6. promote creativity.

The following section lists some important mathematical concepts and terms that can be developed using manipulative aids with gifted students.

Concepts

Mira—symmetry, flip, rotation, perpendicular, bisect

Tangrams—congruent, similar, acute, obtuse, right

Cuisenaire rods—prime, composite, factor, multiple

Attribute blocks—set, union, intersection, subset, attribute

Polyhedra—faces, edges, vertices, surface area, volume

The following sections suggest, for several learning aids, a typical question that might be asked to extend the thinking of mathematically gifted students. Actually, an entire unit might evolve around each suggestion.

Spatial visualization

Tangrams—"Can you make a parallelogram that is not a rectangle using four of the pieces? Five? Six?"

Pattern blocks—"Can you make a design with exactly three lines of symmetry?"

Interlocking cubes—"How many cubes would it take to make an empty cube with four blocks forming an edge? Six? Twenty?"

Patterns and relationships

Attribute blocks—"Can you make a circle of ten blocks, each with two value differences from the adjacent blocks?"

Dominoes—"Your set of dominoes has one missing. Can you find it?"

Tangrams—"Make a design. If the square has a value of 1, what is the value of your design?"

Cuisenaire rods—"How many different trains can you make if you have three rods of different colors? Four? Seven? Twenty?"

Multibase blocks—"How could you represent 567 in base four? Base six? Base twelve? Base twenty?"

Problem solving

Tessellations—"Which of the regular polygons will tessellate? Why?"

Attribute blocks—"Can you fill a six-by-six matrix with blocks so that each block has a one value difference from its horizontal and vertical neighbors?"

Pattern blocks—"Each angle of the square measures 90° and each angle of the triangle measures 60°. Can you find the measure of all of the angles in all the pieces?"

Verification

Cuisenaire rods—"Can you show that the sum of the squares of the two legs of a right triangle is the same as the square of the hypotenuse?" "Can you show me that 21 is a factor of 168?"

Geoboard—"Can you show that the area of a trapezoid is one-half of the height times the sum of the length of the bases?"

Mira—"Can you show that this angle has been bisected?"

Creativity

Tessellations and mirrors—"Can you design a linoleum pattern?"

Geoboard—"How many different-sized squares can you represent on a board with 25 nails? 36 nails? 144 nails?"

Pattern blocks—"Design a dodecagon with the blocks. Can you write a computer program in Logo that will duplicate your design?"

Concluding Comments

This article has focused on the use of manipulatives with two groups of special learners: those with learning difficulties and those with special abilities. It has highlighted different choices in concrete aids and, more important, different purposes and procedures for using those aids to nurture the process of learning mathematics in ways each of these groups of students is capable.

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Selection Criteria

By **Michael C. Hynes**

The task of selecting materials for classroom instruction is a crucial one whether the decision involves textbooks (NCTM 1982), software (NCTM 1984), or other teaching aids. This article focuses on a variety of criteria for the selection of manipulative materials. Let's first begin with a definition:

> Manipulative materials are concrete models that incorporate mathematical concepts, appeal to several senses, and can be touched and moved around by students.

This definition implies that students will have materials in their possession to manipulate (Cathcart 1977). It is not sufficient for students to observe a demonstration of the use of an aid. The act of manipulating the aid allows students to experience the patterns and relations that are the focus of mathematics (Adler 1966).

Little disagreement seems to exist among mathematics educators that experiencing ideas on the concrete level is very important to the learning of mathematics, since mathematics deals with abstractions. A major goal of mathematics instruction is to help children learn to operate efficiently at the abstract-symbolic level with an understanding of the concepts or skills in question. The purpose of using manipulatives is to assist students in bridging the gap from their own concrete environment to the abstract level of mathematics (Fennema 1973). It is the responsibility of the teacher to use these concrete models to foster understanding of the abstract, relate pictures of the models to aid in the movement toward the abstract, and provide practice at the abstract level to insure mastery of a concept or skill. Clearly, manipulative materials are a means to an end (Cathcart 1977; Reys 1971; Sowder 1976). The task of selecting manipulatives becomes a response to the question, "What are the important characteristics of effective manipulative materials?" The following pedagogical and physical criteria should be considered when selecting manipulative materials for a particular purpose.

Pedagogical Criteria

1. *Clear representation of mathematical ideas*

Manipulatives can help children understand mathematical concepts only if the material selected clearly represents the idea in question. The representation of an idea should be natural and easily understood (Sowder 1976). Many contrived models of multiplication of integers have appeared in the literature, but few, if any, manipulative materials offer a student a clear picture of why a "negative times a negative is a positive."

If the goal is to teach subtraction with regrouping, materials that can actually be regrouped are mandatory.

2. *Appropriateness for students' developmental level*

It is important to select materials so that all children can learn using an aid appropriate for their developmental level and learning style (Reys 1971). A critical concern is providing for the wide diversity of individual differences among students (Fennema 1973). Some students learn best while attending to a particular attribute of a manipulative, say length, whereas others may not grasp the significance of the relative length of rods, rows of cubes, or sticks. Attribute blocks usually include the attributes of size, shape, color, and thickness. Cuisenaire rods capitalize on the use of color and length; the focus of the geoboard is primarily on length and area. It is important that manipulatives be selected on the basis of the children's ability to operate effectively with the apparent attributes of the aid.

Materials selected for use must also be consistent with the motor development of the child. For example, a young child may have difficulty manipulating such materials as centimeter cubes, which are very small. The use of a larger cube can be more appropriate in such a situation.

3. *Interest*

Manipulative aids do arouse students' interest and improve their motivation to learn mathematics (Fennema 1973). This motivation is often elicited by the physical characteristics of the materials that will be discussed later. Furthermore, interest seems highest

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Table 1
Manipulative Material Checklist

Manipulative: _____ Supplier: _____ Cost: _____

Concepts developed or reinforced _____

Grade levels: _____

	Poor	Below average	Average	Above average	Excellent
Pedagogical criteria					
1. Clear representation of mathematical ideas					
2. Appropriateness for students' developmental level					
3. Interest					
4. Versatility					
Physical criteria					
1. Durability					
2. Simplicity					
3. Attractiveness					
4. Manageability and ease of storage					
5. Reasonableness of cost					

when each child can take part in the lesson by manipulating the aids.

4. *Versatility*

Some materials stimulate a number of senses at the same time and can be used to teach many mathematical concepts at various grade levels. For example, many materials can be used to teach whole-number, rational-number, and geometric concepts. Grossnickle et al. (1983, 37–39) provided some useful information on the versatility of several manipulatives: abacus, attribute blocks, base-ten blocks, geoboards, pattern blocks, Unifix cubes, and place-value pocket charts.

Physical Criteria

Pedagogical concerns are important in the selection of manipulative materials, but a consideration of the physical characteristics of manipulatives is also crucial. Information about the physical appearance of materials is often available in reviews in the *Arithmetic Teacher* and in commercial catalogs and brochures. In reading this literature, you should note the durability, simplicity, attractiveness, ease of manageability, and storage requirements, and, of course, the cost of the materials.

1. *Durability*

Manipulatives are often costly investments. Thus, the materials should be made to withstand normal use and handling by children (Reys 1971; Grossnickle et al. 1983). Materials that seem too fragile for use by children are of little value. Remember, it is imperative that students actually manipulate the materials.

2. *Simplicity*

A manipulative should be easy to operate. The complexity of its construction should not distract or confuse a student. Because of the desire to buy aids that are multipurpose, care must be exercised to select simple devices that clearly portray concepts being taught.

The simplicity of a manipulative is directly related to the complexity of the mathematical idea being modeled. It seems, however, that the simpler models of even more complex ideas are the ones most easily remembered (Dewdney 1985).

3. *Attractiveness*

Attractiveness is one key to increasing students' motivation and interest. The manipulatives should be bright, precisely constructed, and aesthetically pleasing (Hamilton 1966; Reys 1971). The appearance of the aid should not distract the student from the concept to be learned, however. Periods of free play with the manipulatives, before mathematics instruction, can help insure that students are attentive to the modeling of mathematics and not the aid itself.

4. *Manageability and ease of storage*

Careful plans must be made for the distribution and collection of materi-

als during a lesson. Some aids are packaged for use by individuals or small groups so they can be distributed and collected with a minimum of time and disturbance. Other aids can be kept by students at their desks for as long as necessary. The noise caused by the manipulation of materials can be reduced by putting a piece of felt or flannel on each desk to dampen the sound.

Storage must also be considered. If a manipulative aid has many small parts, they can easily be lost if adequate storage is not available. The teacher should look for materials that can be easily and securely stored in the classroom. Those aids that must be stored outside the classroom tend not to be used.

5. Reasonableness of cost

When pricing aids, consider their use, versatility, and life expectancy. An aid that can be used to teach many mathematical concepts over many years may be less costly than disposable activity sheets. Some indirect costs, like periodic maintenance, must also be considered. Some pieces of a set of manipulatives may be lost and require replacement. In-service training of teachers may be necessary.

Not all manipulative materials must be purchased. Some can be made by the teacher or the students. Bean sticks, pegboards, and geoboards are examples of aids that can be made in the classroom. In addition, see the "Let's Do It" sections in past issues of the *Arithmetic Teacher*. If materials are homemade, the quality of their construction must not have a negative effect on learning. When children make a set of bean sticks to show base-ten place value and only put eight beans on a stick, the aid will not be of much value. Likewise, geoboards with a rectangular rather than a square array of nails will detract from the effectiveness of this versatile aid. Generally, the selection criteria previously discussed also apply to teacher- and student-made manipulative aids.

Conclusion

Selecting manipulatives for use in mathematics instruction is an important responsibility of teachers. Obviously, the effective use of the manipulatives chosen is necessary for children to gain an understanding of abstract concepts. Clearly, the selection process is only the first step in helping children understand mathematics. To assist in this first step, table 1 offers a framework for decisions in choosing manipulative materials.

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MATHEMATICS STAFF DEVELOPMENT
1987-1988

GRADES 4 AND 5

MANIPULATIVE MATERIALS KIT

VENDOR: CUISENAIRE

DESCRIPTION

A Collection of Math Lessons From Grades 3 Through 6
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Fraction Bars-Starter Set
Fraction Bars Transparencies
Decimal Squares Starter Set
Overhead Decimal Squares
Probability Spinners with Color Divisions
Probability Spinners with Numeral Divisions

ORGANIZING FOR EFFECTIVE INSTRUCTION

THE ROLE OF THE TEACHER IN USING MANIPULATIVES

* to select the appropriate manipulative to be used in light of the needs of the student and the mathematics objective(s) to be learned

* to guide and direct the learning activities by using good questioning techniques

* to ask questions which stimulate thinking

* to ask "how" questions that encourage the students to manipulate physical objects in order to model their answers

* to allow the student to discover as much as he/she can for himself/herself

* to assist students to share and clarify their thinking and arrive at "discoveries" through the use of concrete materials

* to ensure that each child works at his/her highest level of ability

* to establish a classroom management system for the distribution, use and collection of manipulatives

* to encourage sharing and distribution of manipulatives among teachers

* to conduct staff development workshops for colleagues at grade conferences, faculty conferences....

* to disseminate a philosophical approach to teaching/learning mathematics with manipulatives to parents of your students

* to maintain an inventory of classroom manipulative materials

Management Suggestions for Using Manipulatives

1. Pose a question, situation, or problem.
2. Explain the purpose of the manipulative.
3. Demonstrate the manipulative. Model its use by "talking through" what you are doing and what you are thinking.
4. Focus on mathematical vocabulary and concepts. Write key words or phrases on the chalkboard.
5. Encourage students to question you on what you are doing and why you are doing it.
6. Encourage students to suggest other methods of solving the problem.
7. Divide the group into small groups or partners.
8. Circulate among the students. Assume the role of a coach.
9. As they work with the manipulative, encourage students to verbalize what they are doing and why. Expect a little noise.
10. Let students summarize their findings.

Organizing for Mathematics Instruction

By **Carol Novillis Larson**

A plan that incorporates diagnostic teaching, unit planning, and flexible grouping is one viable alternative for organizing the elementary school classroom for mathematics instruction. I used this plan when I taught fifth and sixth grades in New Hampshire, and some Title I third- and fifth-grade teachers in Tucson, Arizona, have used it during the past two years. It takes into account three important variables in teaching mathematics:

1. The individual differences among children in mathematical ability and achievement
2. The content of the mathematics curriculum
3. The limited amount of time available to teach mathematics

The importance of each of these three variables is discussed in turn. What follows is a description of an organizational plan that can make it possible for each student in your classroom to become successful in learning mathematics.

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Individual Differences

In a typical elementary school classroom, twenty-five to thirty students are assigned to a teacher. These children are usually similar only in chronological age. They differ greatly in their achievement in mathematics, their mathematical ability, and their attitudes toward mathematics. These differences in achievement increase with age, and so a greater variety occurs in the upper elementary school grades than in the primary school grades. Achievement in mathematics varies for individual students depending on the different mathematical content and processes. For example, it is possible for a child to have skills in whole-number operations but have little knowledge of fractions and decimals. Therefore, it is advantageous to group children on the basis of their knowledge of a particular mathematical topic and to regroup when beginning a new topic.

Mathematics Curriculum

The National Council of Supervisors of Mathematics (NCSM) in its "Position Paper on Basic Skills" (1977) identified ten basic-skill areas that need to be taught to all elementary school children. Some of the basic-skill areas are concerned with content strands and some with mathematical processes. Figure 1 is a content-process matrix that was developed on the basis of NCSM's position paper (1977) and the National Institute of Education's (NIE) Euclid conference (1975). Most of the mathematics taught in the elementary school can be included in at least one cell of this matrix. This matrix also suggests skills that are not generally emphasized in the elementary school, such as problem solving in geometry and estimating with fractions and decimals. The matrix helps to make clear

Fig. 1 A content-process matrix for curriculum topics

Content-process matrix	Whole numbers	Rational numbers		Geometry	Measurement	Statistics and probability
		Fractions	Decimals			
Knowledge						
Estimating						
Computational and other skills						
Problem solving						
Application						

the extent of the content and processes children are expected to master in the elementary school; a lot is to be learned.

Given the differences among students, it is not reasonable to expect all children to master the same curriculum by the end of sixth grade. Some children have more ability than others; these children will and *should* learn more. So the teacher needs to make a decision about what to teach different students. Teachers sometimes decide that whole-number computation is the most important area of the curriculum and that children cannot be exposed to anything else until they have reached some specified level of mastery for their grade level. This decision results in some children in primary grades spending the greater part of their three years of mathematics instruction learning about place value and the addition and subtraction of whole numbers. These children are not given the opportunity to develop beginning fractional and geometric concepts. This approach has far-reaching consequences for what occurs when these children reach the intermediate grades.

All students should study all strands in the curriculum each year.

Another decision that teachers sometimes make is to group children permanently on the basis of their overall mathematical achievement and ability. The teacher then varies the pace at which each group progresses through the textbook that is being used. The children in the group paced fastest usually study all the various topics covered in the textbook, whereas the students in the group paced slowest usually get bogged down on some topics and are never exposed to content strands that come later in the book.

Consider the possible impact of these practices on two imaginary fourth-grade students, Leslie and Pat. Leslie is in the slowest-paced group and never studied fractions in the primary grades. Pat, who is in the fastest-paced group, was introduced to fractions in the first, second, and third grades. Pat, who has more natural ability in the area of mathematics, has

had the opportunity gradually to develop elementary fractional concepts over a three-year period. In fourth grade, both students now face the two or three chapters in their textbook on fractions, equivalent fractions, and addition and subtraction of fractions. Leslie will be exposed in a few weeks' time to the fractional concepts that Pat had three years to master, and then Leslie will also be expected to master the "new" fourth-grade topics. This expectation is not fair to Leslie. That is, Leslie should also have been given the opportunity to begin to develop fractional concepts in the primary grades.

Teachers need to make a different decision about what mathematics to teach. Each child needs to be taught appropriate mathematical content and skills in *each* major strand, *each* year. This goal can be achieved by acknowledging that each child can learn something new about each major strand in the curriculum regardless of his or her progress in other strands. Regrouping in addition and subtraction are not prerequisite to understanding the meaning of multiplication and division. Very few skills with whole numbers are prerequisite to beginning work with fractions, and none are necessary for exploring many geometric concepts.

This choice of teaching each child something new about many different mathematical topics can be implemented through a unit approach to teaching mathematics. In such an approach each child in the classroom studies the same mathematical strand at the same time, but at his or her own level of ability and achievement. This feature is a vital component of the organizational plan that will be described later in this article.

Time

There is a limited amount of time in which to teach mathematics to your students. We often think of having an entire year in which to teach the mathematics curriculum at each grade level. Actually, how much time do we have in that year? Teachers usually plan to teach mathematics for anywhere from twenty minutes to an hour a day. This is allocated time. A recent study reported by Berliner (1980), however, shows that allocated time does not always equal instructional time. In some classrooms, a great deal of time is lost in transition and in behavioral management.

Most school years are about 180 days, but certain factors, such as teacher absences, holiday programs and parties, field trips, and standardized testing, must be considered in estimating the number of instructional days. Berliner (1980) estimates the "functional" school year at about 150 days. So teachers allocate from 50 to 150 hours a year for mathematics instruction.

Unit planning allows students to be taught many topics.

The elementary school teacher needs to face this problem of a limited amount of time and must make decisions about how best to use the time available. In consideration of the need to plan for individual differences and to teach all children something about each content area, one possibility is to assign a certain number of weeks to each content area. For example, all children in the class could be taught a three-week unit on geometry. During that three weeks, children would be grouped according to their present knowledge of geometry.

Table 1
A Year's Schedule of the Mathematics Curriculum for Grade 3

Strand	Number of weeks		Dates
Numbers and numeration	3		8 Sept.–26 Sept.
Addition and subtraction of whole numbers	9	4	6 Oct.–3 Nov.
		5	10 Feb.–17 March
Multiplication and division of whole numbers	8	4	5 Jan.–26 Jan.
		4	28 Apr.–26 May
Measurement	3		4 Nov.–26 Nov.
Time	2		9 Apr.–27 Apr.
Money	2		27 Jan.–9 Feb.
Geometry	3		1 Dec.–19 Dec.
Beginning fractions	3		17 Mar.—8 Apr.
Graphing and tables	1		29 Sept.–3 Oct.
	All year		
Problem solving	All year		
Total	34		

Planning for the Year

The first task that faces the teacher in implementing a system of flexible grouping and unit planning is to plan a year's schedule of the mathematics curriculum at the beginning of the school year. This task can be done by listing the major topics that are usually taught at that grade level. The major broad topics usually taught in grade 3 are listed in table 1. Be optimistic and plan to teach mathematics for about 34 weeks, or 170 days. The 34 weeks are then assigned to the topics depending on the emphasis placed on each topic at that grade level. If more than 4 or 5 weeks have been assigned to a topic, then it is divided into 2 or 3 periods or units. For example, in table 1, 9 weeks have been assigned to teaching the addition and subtraction of whole numbers. Some teachers prefer to teach a 4-week unit on this topic in the fall and another 5-week unit after winter vacation. Other teachers prefer to teach three 3-week units on this topic.

The next step is to assign tentative dates to each topic using the year's school calendar and taking holidays and vacations into account. The dates in table 1 are based on one school district's calendar for 1980–81. Many different sequences of the topics are possible. Not all third-grade teachers agree on how many weeks to assign to each topic. Therefore, time assignments as well as sequences will vary with individual teachers.

Table 2
Group Schedule for Week of 27 April

Day	Time	Sandy's group	Bernard's group	Paul's group
Monday	25 min.	Instruction	Follow-up activity	Laboratory activity
	25 min.	Follow-up activity	Games	Instruction
Tuesday	25 min.	Games	Instruction	Follow-up activity
	25 min.	Instruction	Follow-up activity	Games
Wednesday	25 min.	Follow-up activity	Laboratory activity	Instruction
	25 min.	Laboratory activity	Instruction	Follow-up activity
Thursday	25 min.	Laboratory activity	Follow-up activity	Instruction
	25 min.	Instruction	Games	Follow-up activity
Friday	50 min.	Large-group instruction		

Grouping

A teacher using this year's schedule would teach numbers and numeration to all third-grade students from 8 September to 28 September. Since all students do not have the same knowledge of this topic, it may be necessary to administer to the whole class a diagnostic test on the concepts and skills in this topic (see McKillip [1979] for a good description of how to write teacher-made tests). On the basis of the results of the test, the teacher would then form two, three, or four groups of students who had about the same knowledge of numeration. These groups would stay together only for the duration of this specific unit. Since all the children are working in the same general area, it is very easy to move students to a different group if they were originally placed in an inappropriate group or if they are making more progress than the other students in their group.

Planning for Different Groups

One concern teachers might have when they first begin to group for instruction is what the other two or three groups will be doing while one group is being taught by the teacher. The teacher must plan independent activities at the levels of the different groups that are consistent with and reinforce the instruction that the teacher is giving the different groups. If one group is being taught to model two- and three-place numbers with concrete materials, to write these numbers in expanded notation, and to indicate the ones, tens, and hundreds places of numbers, then all independent activities for this group should be designed to reinforce or extend these objectives.

Flexible grouping allows students to work at their own level.

If one hour a day is allocated to the teaching of mathematics, then it is possible to plan two twenty-five-minute instructional periods in that hour. The other ten minutes is taken up in transition time. Thus, if you have three groups, you can teach a lesson to two groups each day. When teaching lessons to each group, the emphasis should be on developing meaning for the mathematics being studied. The teacher should make use of many different concrete models and relate the mathematics as much as possible to the daily life of the students. Students should be encouraged to ask questions and to express their feelings and ideas related to the mathematics being discussed.

While you are teaching one group, the other two groups are working independently on follow-up activities, games, and laboratory activities. At the end of each lesson the students are assigned a follow-up activity that reinforces the lesson. It usually consists of exercises in the textbook or workbook that are directly related to the lesson. If the instruction occurred at the end of the mathematics period, then the follow-up activity is scheduled for the first half of the mathematics period on the next day.

Games and puzzles are used to reinforce the skills and concepts that are the focus of previous lessons with a particular group. The aim is to help children use mathematical facts and algorithms in a fun context. Various kinds of games can be used: card games, board games, dice games, and forms of bingo. The *Arithmetic Teacher* is an excellent source of games (see the index that appears in the May issue each year).

Laboratory activities are more open-ended. Their purpose might be exploration that will lead to readiness for a future lesson, reinforcement or application of a previous lesson, problem solving in the area of study, or enrichment. In most laboratory activities, students generate their own data and record their findings in tables, graphs, or other appropriate modes (see Biggs and MacLean [1969] and Reys and Post [1973]).

Table 2 is a schedule for one week for a class with three groups. In this particular week, the teacher has planned a large-group activity on Friday. Large-group activities are possible in this system as all students are working in the same general content area. A large-group lesson might focus on solving problems that can be solved in many ways and at different levels.

At any given time, the teacher is teaching one group, another group is engaged in a follow-up activity, and the third group is playing games or working on laboratory activities. Only one group is usually playing games or working on laboratory activities because these activities tend to be noisier. Usually each group is involved in two different types of activities during a mathematics period.

Many teachers agree that this plan sounds like a viable way to organize mathematics instruction but at first may be overwhelmed by the amount of planning and preparation involved. A suggestion to these teachers is to begin gradually with only some aspects of the plan. As they become more at ease with the new things they are doing, they can add other aspects of the plan in subsequent units. A teacher might separate the class into only two groups and experiment with planning two different lessons and follow-up activities a day. Then, in the next unit, games might be used at least once a week and one or two laboratory activities might be tried. The experience gained working with two groups for three or four units will make working with three groups a natural step.

Management

How can a teacher keep three or four groups working smoothly? Student leaders, written communications between the teacher and individual students, and peer tutoring have been used together successfully. Each student is given the opportunity of being a group leader for the duration of a unit during the school year. The groups are given the names of their students leaders (see table 2). The student leader for each group distributes and collects papers, folders, games, laboratory cards, and materials.

To organize students' papers and as an aid to communication, each student is given a folder in which to place all written work. All written work is placed in the folders at the end of each mathematics period, and later the teacher can read the contents of each folder. A blank piece of paper is stapled on the inside cover of each folder to facilitate communication between the teacher and students concerning their work. The teacher can write encouraging messages to students and can make individual changes in group assignments on this sheet. Students can also write on these sheets to tell the teacher that they need extra help, to ask questions, or simply to write anything about their learning of mathematics. Children do enjoy writing to the teacher and getting responses.

Peer tutoring can be used when a student needs some individual assistance. Tutors are always members of the same group as the child needing assistance.

Evaluation

When children are taught in groups, they should be tested only on their group's objectives. If grades are used, students should be graded on the work done during the unit and on the results of tests given to test the objectives that were taught to their group. The grades on report cards should then reflect the grades given in class during the unit. Students' grades should be based on how well they learned the mathematics designated as appropriate for them.

Diagnostic teaching helps a teacher to plan instruction.

Students' Awareness

An important component of this organizational system is the students' comprehension of what is going on. In order to help students understand the purposes for grouping, you may wish to spend several class periods discussing individual differences. Students need to understand the importance of personal progress in mathematics and to realize that their success should be measured by their own progress, not by comparing themselves to others. At the end of a unit, students should be asked to identify what they have learned to help them see their personal progress. What do they now know that they didn't know three or four weeks ago?

Throughout the year, it is important for the teacher to be honest with students, openly to face the difficult questions and problems that arise, and to discuss them with the students involved. Also, the members of each group should be aware of their group's objectives. A great deal of emphasis needs to be placed on children taking more responsibility for their own learning.

Conclusion

This article described an organizational plan that incorporates flexible grouping, unit planning, and diagnostic teaching. Flexible grouping allows students to work at their own level. Unit planning makes it possible for each child to be taught many different mathematical topics. Diagnostic teaching using pretests and observation helps teachers identify each student's knowledge of a mathematical strand and the instructional level at which that student should be placed.

Teachers who have implemented this plan say that they teach more mathematics to each student and that their classroom is a busy, happy place.

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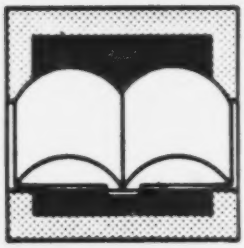


Research Report

Individualized or Cooperative Learning

By **Marilyn N. Suydam**
Ohio State University
Columbus, OH 43212

From a recent study comes the conclusion that individualization of mathematics instruction using programmed materials with individuals or cooperative learning teams can be effective and manageable by a single teacher without an aide (Slavin, Leavy, and Madden 1984).

When teams were used, the following procedures were adopted:

—Four or five students with different abilities worked together. (Every four weeks, new teams were formed.)

—Scores and recognition were given to teams on the basis of the speed and accuracy of the group's members. (Thus, students were motivated.)

In both individual and team situations, these steps were followed:

—Diagnostic tests were used to help place students in the program.

—Curricular materials consisted of a step-by-step instructional sheet, several skill sheets with twenty problems for each subskill, two quizzes, a final test, and answers.

—Students each did four problems, then checked their own work or had a team partner check the answers. If correct, the students went on to the next skill sheet; if incorrect, they had to try the next four problems, and so on. Answering four problems in a row correctly on the last skill sheet meant that a quiz could be tried. Thus, students managed the individualized program themselves, freeing the teacher for instruction.

—The teacher reviewed lessons daily, with one to ten students who were at about the same point in the curriculum, going over troublesome points and preparing students for upcoming units.

Both the individual and team modes resulted in mathematics achievement of students that was significantly higher than that of students given whole-group instruction. The key variables appeared to be the following:

- Self- or partner-checking
- Frequent mastery checks allowing students to skip material that they already understood

Reference

Slavin, Robert E., Marshall B. Leavy, and Nancy A. Madden. "Combining Cooperative Learning and Individualized Instruction: Effects on Student Mathematics Achievement, Attitudes, and Behaviors." *Elementary School Journal* 84 (March 1984):409–22. ■

PROFESSIONAL DATES

NCTM 63d Annual Meeting
17-20 April 1985, San Antonio, Tex.

NCTM 64th Annual Meeting
2-5 April 1986, Washington, D.C.

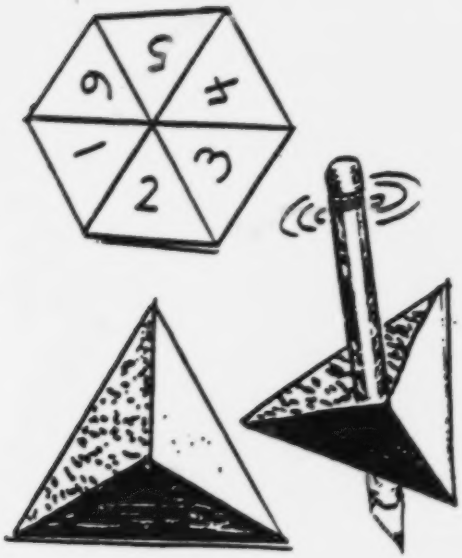
NCTM 65th Annual Meeting
8-11 April 1987, Anaheim, Calif.

For a listing of local and regional meetings, contact NCTM, Dept. PD, 1906 Association Dr., Reston, VA 22091. Telephone: 703-620-9840; CompuServe: 75445,1161; The Source: STJ228.

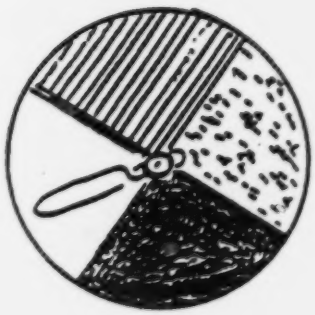
Children solve the problem of sharing six cookies among the four of them.

DEVELOPING INSTRUCTIONAL APPROACHES FOR TEACHING PROBABILITY

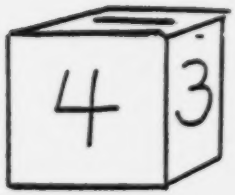
SUGGESTED DEVICES FOR THE STUDY OF PROBABILITY



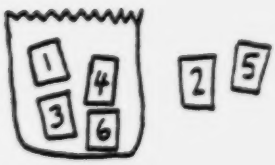
Cut a hexagon or triagle from cardboard. Push a pencil through the center. Give it a spin!



Use a paper clip and a paper fastener to make a spinner.



Write the numerals one to six on each face of a cube.



Draw numeral cards from a paper bag

Research Report

Learning Probability Concepts in Elementary School Mathematics

Many curriculum groups and individuals have recommended the inclusion of topics in probability in the elementary school mathematics program. In "A Case for Probability," an important position statement in this journal, Jones (1970) recommended probability because (*a*) the subject deals with ideas and patterns that grow over time; and (*b*) evidence suggests that appropriate units can be taught in the elementary school.

Piaget's pioneering studies of the ideas of chance (Piaget and Inhelder 1975) are the foundation of research in this area. On the basis of his explorations, he divided the development of probabilistic thinking in children into three age-related stages. In stage 1 (up to age 7 or 8), the student does not understand random phenomena but continually looks for a hidden order. The ability to make predictions is unstable, depending on preference, on expectation of an outcome to "catch up," or on observation of the largest frequency to date. "Miracles" (extremely unlikely events) cause no surprise. In stage 2 (up to age 11 or 12), a global understanding of randomness is achieved, but not of the effects of large numbers. "Miracles" cause a search for an underlying cause. In stage 3, formal thought, the law of large numbers is understood and probabilities are successfully assigned.

Since Piaget's work, a number of studies have dealt with the study of probability in the elementary school. These studies tend to fall into one of two types: (1) What do elementary school students know about probability prior to formal instruction? (2) What topics in probability can elementary school students learn?

1. *What do elementary school students know about probability prior to instruction?*

Work with younger students using simple probability devices, by Perner (1979) and by Falk, Falk, and Levin (1980), found that the ability to predict the most probable outcomes improves with age, beginning at about age 6. In grades 2 and 4, McLeod (1971) investigated a number of topics, including reasoning from distributions, independence, and the law of large numbers. He found that most students were able to apply the concepts prior to study.

Schroeder (1983) found that students in grades 4–6 had notions of probability, chance, and relative frequency. Doherty (1965) looked at several formal probability topics in the same grade levels and found that older students and those with a higher level of arithmetic achievement scored higher. In contrast, Mullenex (1968), who worked with grades 3–6, concluded that age, general ability, basic vocabulary skill, and work-study skill are not relevant to understanding randomness, assigning probabilities to events, or recognizing the long-range stability of random phenomena.

Students' misconceptions about various aspects of probability are discussed in Falk, Falk, and Levin (1980); Green (1983a, 1983b); Dunlap (1980); Armstrong (1972); and, of course, in Piaget and Inhelder (1975).

Common misconceptions mentioned include counting the number of desirable outcomes rather than considering the ratio of desirable to undesirable outcomes; expecting all outcomes in an experiment to be equally likely; holding a bias against certain numbers (e.g., believing that it is hardest to throw a 6 on a die); selecting on the basis of preference (e.g., color) rather than on probability (this is particularly true of young children); and failing to make inferences from data.

2. *What can elementary school students learn about probability?*

Cantor, Dunlap, and Rettie (1982); Jones (1974); Dunlap (1980); and McLeod (1971) used various types of probability devices with children and found that they learned. Jones found better learning if the devices show the unit (e.g., on spinners). Cantor, Dunlap, and Rettie (1982) recommend that instructional tasks and criterion tasks be kept the same (i.e., don't expect much transfer).

Hirst (1977), Schroeder (1983), and Armstrong (1972) support the use of experiments to teach probability. Armstrong cautions that units of two-and-one-half to three weeks in length are tiring. Doherty (1965) found that students learned more about probability *while taking* a test on the topic. Gipson (1971) taught some students in grades 3–5 and determined that they could learn the topics but had difficulty counting and listing all the possible outcomes of an experiment.

Shepler (1969, 1970) used mastery-learning techniques with sixth graders and found that they mastered ten of fourteen objectives related to probability, counting procedures, cumulative frequency graphs (showing experimental probabilities stabilizing), and the law of averages. The retention study reported by Romberg and Shepler (1973) demonstrated continued high-level performance four weeks later.

Thus the research indicates that students have some understanding of probability and related topics, that this understanding increases with age and instruction, and that probability can successfully be taught in the elementary school in carefully selected experiments. The curriculum groups

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and individuals recommending that probability be a topic for the elementary school are thus on target.

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Unit 33

4.12.03 PROBABILITY

(1-2 lessons)

PERFORMANCE OBJECTIVES

The student will be able to...

*4.12.03.01A determine probabilities by comparing favorable outcomes with total possible outcomes and express as fractions;
e.g.,

This spinner has 3 equally likely outcomes. Since 1 out of the 3 outcomes is red, the probability of spinning red is 1 out of 3 or $\frac{1}{3}$.

4.12.03.02A determine probabilities where the outcomes are not equally likely and express as fractions;
e.g.,

This spinner has 4 equal sections. Three of the sections are green. The probability of spinning green is 3 out of 4 or $\frac{3}{4}$.

4.12.03.03A predict the outcomes of probability experiments.

e.g.,

If we spin this spinner 30 times, how many times do you think the spinner will land on red?

4.12.03

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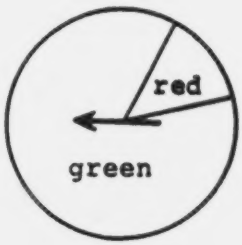
VOCABULARY

Terms

chance	outcome
equally likely	predict
favorable	probability

GETTING STARTED 1:

Prepare and display a spinner and a tally chart.
e.g.,



Color	Tally of Outcomes
red	
green	

Ask each student to spin the spinner and record the outcome with a tally mark on the chart. When the chart is completed say, "Which color came out more often? (Green came out more often.) How many more times did green come out than red? (Answers will vary depending on tallies.) Why do you think green came out more often? (There was a much larger green area than red area on the spinner.) When one outcome has a better chance of happening than another, we say it is the outcome that is more likely to happen."

Repeat the experiment a second time and compare the outcomes again. Say, "What did you notice? (Green came out more often than red again.) Using this spinner, green will almost always come out more often than red. It is possible for red to come out more often than green, but it is not very likely to happen."

GETTING STARTED 2:

Prepare and display 5 red blocks, 5 blue blocks, and a paper bag. Place 5 red blocks and 1 blue block in the bag and say, "If I asked you to pick a block out of this bag without looking, could you be

GETTING STARTED 2: cont'd

certain that you would pick a red block? (No.) Why not? (If I can't look in the bag when I pick, I might pick a red block or a blue block. I can't be sure which color I will pick.) Which color do you think has the better chance of being picked? (A red block has a better chance of being picked.) Why?" (There are more red blocks than blue blocks in the bag.)

Repeat this procedure with 5 blue blocks and 2 red blocks and again with 5 red blocks and 5 blue blocks.

4.12.03

4.12.03.01 **determine probabilities by comparing favorable outcomes with total possible outcomes and express as fractions**

STRATEGY 1: Prepare and display a spinner with three equal sections, each a different color.
e.g.,

Say, "When we spin this spinner, what outcomes are possible? (It is possible for the spinner to land on red, blue, or green.) What do you notice about the size of each colored area? (Each colored area is equal in size.) Since this spinner has been divided into 3 equal parts, what fractional name can we give each part? (Each part is one third of the spinner.) Does any one color, or outcome have a better chance of occurring than the others? (No.) Why not? (All the possible outcomes are the same size on the spinner.) Let's find the probability of the spinner landing on red. Finding the probability means finding how likely it is that an event will happen. We will call red the favorable outcome because it is the outcome we want to happen. How many places are there for the spinner to land on red?" (There is 1 red area on which the spinner can land.) Write the following on the chalkboard:

number of favorable outcomes = 1

Say, "How many possible outcomes are there?" (3 outcomes are possible; red, blue, and green.) Write the following on the chalkboard:

$$\frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{1}{3}$$

STRATEGY 1: cont'd

Say, "The probability of the spinner landing on red is 1 out of 3, or $\frac{1}{3}$. What do you notice about $\frac{1}{3}$? (It is the same number as the fractional part of the spinner that is red.) What will be the probability of the spinner landing on blue?... green?" (The probability of the spinner landing on blue is $\frac{1}{3}$,... green is $\frac{1}{3}$.) Write the probability of each outcome on the chalkboard and ask the students to write them in their notebooks.

Repeat this procedure with other spinners.

e.g.,

STRATEGY 2:

Prepare and display 5 blocks, each a different color, and a paper bag.

e.g.,

Say, "We will learn how to tell what the chances are of picking one color, such as green, if we were to place all these blocks in the bag and pick one without looking. We call the outcome we are looking for the favorable outcome. The green block will be our favorable outcome. How many green blocks (favorable outcomes) are there?" (There is 1 green block.) Write on the chalkboard and say, "There is 1 favorable outcome."

Say, "How many blocks are there that have a chance of being picked? (5 blocks have a chance of being picked.) We say these are all the possible outcomes." Write on the chalkboard and say, "There are 5 possible outcomes." Draw a line between the 1 and 5 to represent a fraction.

e.g., There is $\frac{1}{5}$ favorable outcome.
There are 5 possible outcomes.

Ask a student to circle the fraction and read it aloud. Say, "The chance of picking the green block

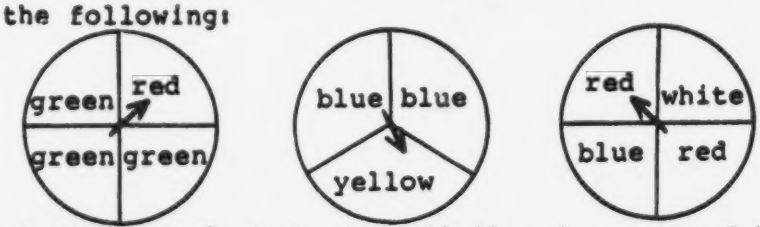
STRATEGY 2: cont'd would be 1 out of 5, or $\frac{1}{5}$. What is the probability of picking the black block? (The probability of picking the black block is 1 out of 5 or $\frac{1}{5}$.) What can you say about each block? (The probability of any one of the blocks being picked out of the bag is $\frac{1}{5}$.) Since each block is the same size, shape, and weight, we cannot tell one block from another without looking at the color. Therefore, we say that each block has an equally likely chance of being picked. That does not mean that it must happen 1 out of 5 times. It is not certain, but we have an idea of how likely it is to happen. How do we determine probability?" (We determine probability by thinking of how many favorable outcomes there are and writing that number as the numerator of the fraction. The denominator is the number of all possible outcomes.) On the chalkboard write:

Probability of an event = $\frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$

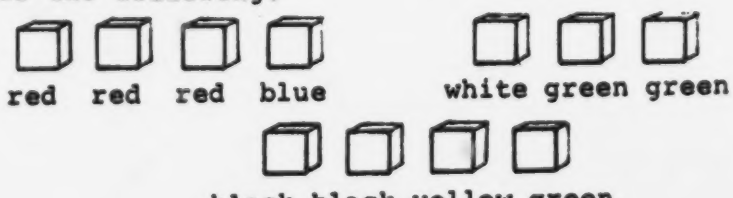
Repeat this procedure with other sets of blocks, e.g., a red block, a blue block, and an orange block.

4.12.03.02 **determine probabilities where the outcomes are not equally likely and express as fractions**

STRATEGY 1: Adapt Strategy 1, P.O. 4.12.03.01 using spinners such as the following:



STRATEGY 2: Adapt Strategy 2, P.O. 4.12.03.01 using sets of blocks such as the following:



4.12.03

4.12.03.03 predict the outcomes of probabililty experiments

STRATEGY 1: Prepare a spinner.
e.g.,

Say, "What is the probability of the spinner landing on red? (The probability of the spinner landing on red is $\frac{1}{3}$.) What does a probability of $\frac{1}{3}$ mean? (A probability of $\frac{1}{3}$ means it is likely that red will come up once out of every three spins." Write the following on the chalkboard:

Spin 1	Spin 2	Spin 3

Ask 3 students to each spin the spinner and record the color on the chalkboard under the correct title. Say, "How many times did red come up?" (Answers will vary.) If red came up exactly once out of 3 spins say, "Do you think this will always happen? Try the experiment again and see if it happens exactly the same way." If red came up 2 or 3 times, or did not come out at all, say, "Why do you think that happened? Does red have to come up once out of 3 tries? (No. There is always the chance that it won't happen.) Even though we can't be certain, knowing the probability of an event happening can help us make a good estimate about how many times it will happen. How many times do you think red will come out if we spin the spinner 6 times? ... 9 times. ... 30 times?" Record the predictions. Ask the students to spin and record the outcome each time on the chalkboard. Compare the results to the predictions.

STRATEGY 2: Prepare and display a set of blocks and a paper bag as in Strategy 1, P.O. 4.12.03.01 and draw a table on the chalkboard.

STRATEGY 2: e.g.,
cont'd

Color	Tally of Times Picked
blue	
green	
yellow	
white	
black	

Say, "If we place all the blocks in the bag, what is the probability of choosing the white block? (The probabililty of picking the white block is $\frac{1}{5}$ or 1 out of 5.) What does that mean? (It means it is likely that white will be chosen once out of every 5 picks from the bag.) It does not have to happen that way, but knowing the chances helps us make a good guess as to the number of times it should happen. If we guess that the white block will be picked 1 time out of every 5 picks, how many times do you think white will come up in 10 picks?" (It will probably come up 2 times out of 10.) Record this and any other predictions on the chalkboard. Place the blocks in the bag. Ask a student to pick a block, record a tally mark in the correct place on the chart, and place the block back in the bag. Repeat this until 10 picks have been made. Say, "We knew that white had 1 chance out of 5 of being picked. From that, we guessed that white might come up twice in 10 tries. How many times did white come up? (Answers will vary.) Was our prediction close to the actual results? Why do you think that happened?" Continue this procedure, making predictions based on probability for how many times white will be picked in 15 picks, 20 picks, and 25 picks. Record the results on the chart and compare the predictions to the results. Say, "Since chance is involved, our predictions may not always be close to the actual results."

Developing Concepts in Probability and Statistics—and Much More

By **James V. Bruni** *and* **Helene J. Silverman**

In recent years school mathematics programs have included an increasing emphasis on probability and statistics. Many of the activities suggested in these programs offer wonderful opportunities to involve students in active learning experiences in which they perform experiments using a variety of manipulative materials (e.g., coins, spinners, dice, and cards). These activities are highly motivational and certainly lend themselves to the use of manipulative materials. However, all too often such activities tend to become supplementary or enrichment experiences that are quite limited in scope and isolated from the rest of the school mathematics program, and the *connection* between using manipulative materials and learning basic mathematical concepts is weak.

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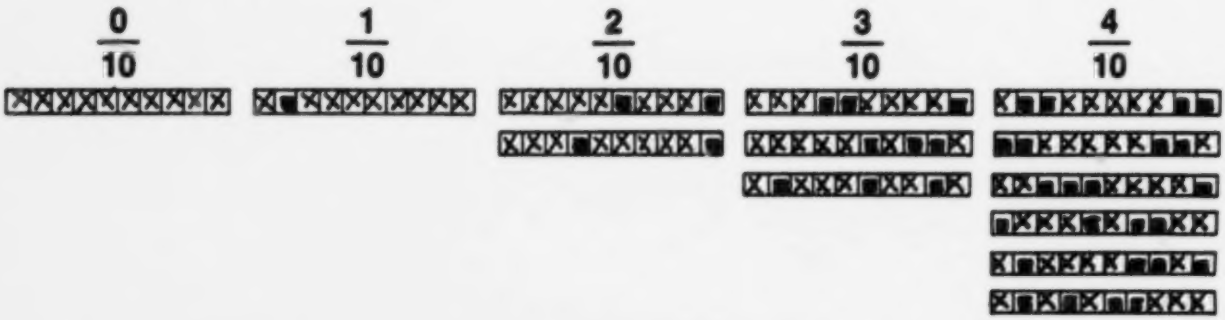
The activities in this article emphasize a four-step approach for using manipulative materials to develop some fundamental concepts in probability and statistics while using ideas about fractions, ratios, proportion, and percentage as an integral part of problem-solving investigations. The four steps are these:

1. Introducing the model
2. Establishing a recording system
3. Reflecting on the experience
4. Generating new experiences

The purpose of this approach is to provide a structured methodology that teachers can use to take greater advantage of the power of manipulative materials. Each experience with manipulative materials leads to a discussion to build necessary vocabulary based on the mental imagery developed (introducing the model); a transfer of the experiences to tables, charts, and graphs as a way of organizing those experiences (establishing a recording system); an examination of the records to recall the concrete experiences, identify possible pat-

Fig. 1 Keeping track of red blocks drawn

Fig. 2 Recording the number of reds in ten trials



terns, summarize information, and raise new questions (reflecting on the experience); and an exploration of related activities intended to reinforce and extend the concepts introduced (generating new experiences).

Primary Grade Activities

Introducing the model

Have students work in pairs. Give each pair a small, brown paper bag and two wooden or plastic cubes of different colors (e.g., red and blue). Have them put the two cubes into the bags. Ask if they can guess which cube they will pick if they take one cube from the bag without looking. Have them take turns guessing, shaking the contents of the bag, and taking a cube from the bag, *replacing it after each turn*. Talk about how good their guesses are. Are they usually right? How often? Which color do they get most often? How often will a specific color be drawn in ten trials? Does a pattern occur in the way a specific color comes up? Have students predict how often a specific color is likely to come up in ten trials and then experiment to check their predictions.

Establishing a recording system

Graph paper cut into strips of ten squares can be used for keeping a record. Give each pair of students a strip of ten squares. Explain that the squares will be used to keep track of how often a specific color (e.g., red) is drawn. Each square represents one turn. Have each pair of students take a cube out of the bag without looking. If it is red, color the square corresponding to that turn. If not, put an X in that square. In this way, each pair of students can keep a record of when a particular color is drawn, how often, and in what order for ten trials (fig. 1). Talk about the recorded results. Did anyone get red *every* time? Did anyone *never* get a red block? Which color was drawn more often?

Each pair of students can repeat this experiment and create records on several strips. You can collect these strips and put them on the chalkboard or bulletin board, sorting them according to how often one color was drawn in ten trials. The notation 0 of 10, or 0/10; 1 of 10, or 1/10; and so on; is useful for describing the results (fig. 2).

The power of manipulatives is enhanced by a four-stage teaching process.

Reflecting on the experience

Discuss the distribution of outcomes the group has generated. How many times did one color usually come up? With a large number of experiments, the results are likely to cluster around the outcomes 4/10, 5/10, and 6/10. Does a "pattern" exist in the order that one color comes up? Do the colors "take turns"? If one color comes up two times in a row, is the next block likely *not* to be that color? As they begin to realize that no constant pattern evolves and that the result of one trial has nothing to do with the next one, they begin to recognize the meaning of a random event and the independence of trials.

Extending the experience

This activity is meant to be just the beginning of many similar probability experiments with bags containing blocks in various color combinations. Encourage students to investigate how likely they are to get a specific color when they change the contents from two blocks of different colors to other combinations of blocks. For example, individual bags might have three or four blocks of two different colors or more blocks of one color than another in varying proportions.

As the students carry out these experiments, have them predict how often one of the colors is likely to come up in ten trials, then complete the experiment to check their predictions. In this way, the class can generate over fifty experiments in a relatively short time. Then discuss the original predictions and the actual distribution of the outcomes. Relate each experiment's outcomes to the contents of the bag. For example, if red usually came up less than half the time, were less than half the blocks in the bag red? As you develop graphs showing the outcomes of several different experiments, you can challenge the students to match these graphs to the original bags.

Intermediate Grade Activities

Introducing the model

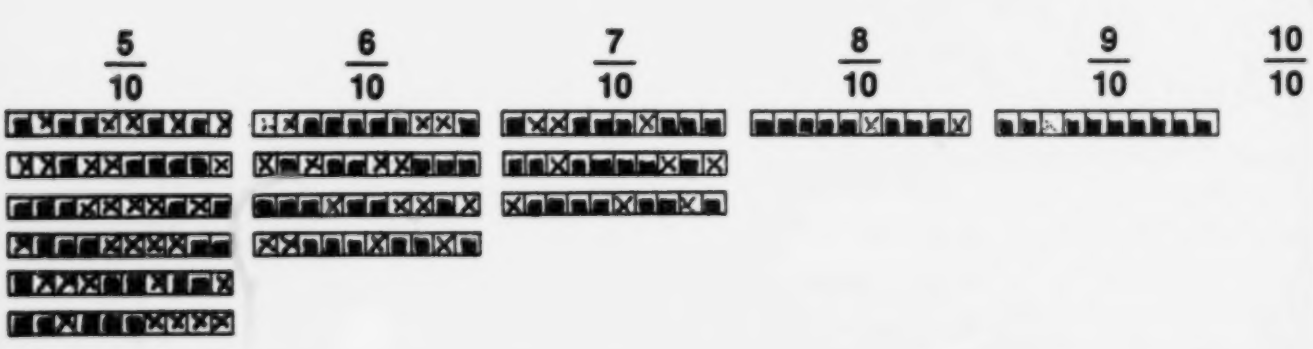


Fig. 3 Extending a table.

Andrea:

Kind	Number
lima	5
kidney	7
pinto	7
black	9
Total	28

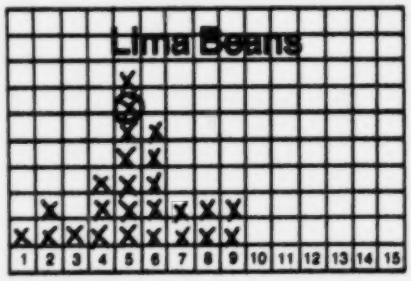
→

Andrea:

Kind	Number	Fractional part	Decimal part	Percentage
lima	5	$\frac{5}{28}$	0.18	18%
kidney	7	$\frac{7}{28}$	0.25	25%
pinto	7	$\frac{7}{28}$	0.25	25%
black	9	$\frac{9}{28}$	0.32	32%
Total	28	$\frac{28}{28} = 1$	1.00	100%

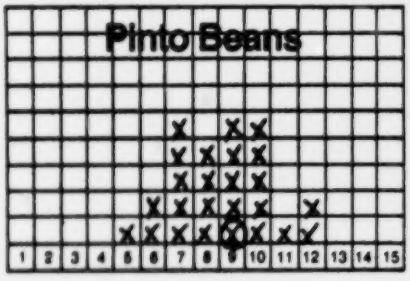
Fig. 4

Number of lima beans in each scoop: 5, 8, 3, 6, 5, 9, 4, 7, 6, 5, 8, 2, 6, 2, 9, 1, 5, 7, 6, 5, 4, 6, 4, 5, 5



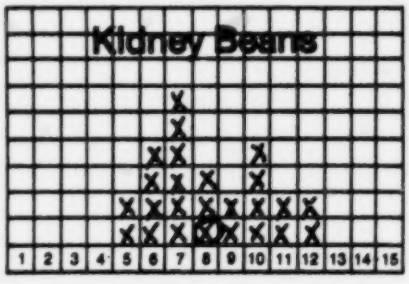
Range 8
Mode 5
Median 5
Mean 5.32

Number of pinto beans in each scoop: 7, 8, 9, 9, 7, 6, 8, 10, 10, 12, 12, 7, 9, 10, 8, 9, 7, 10, 5, 11, 9, 10, 7, 6, 8



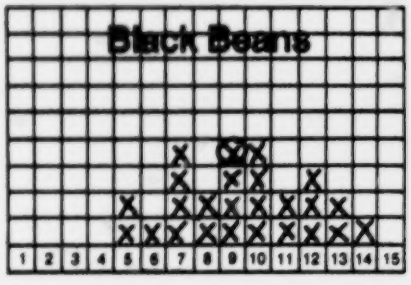
Range 7
Modes 7, 9, 10
Median 9
Mean 8.36

Number of kidney beans in each scoop: 7, 10, 6, 12, 8, 8, 9, 7, 7, 7, 11, 9, 10, 6, 5, 7, 6, 10, 10, 12, 11, 5, 6, 8, 7



Range 7
Mode 7
Median 8
Mean 8.16

Number of black beans in each scoop: 9, 7, 9, 10, 11, 12, 13, 5, 7, 10, 7, 10, 12, 13, 9, 9, 8, 10, 14, 5, 7, 6, 12, 8, 11



Range 9
Modes 7, 9, 10
Median 9
Mean 9.36

Pour the contents of one-pound bags of four types of beans (e.g., lima, kidney, pinto, and black) into a large container and mix them thoroughly. Provide a scoop or spoon so that each student can scoop some beans into a small container (a coffee scoop works well). Before the students begin, ask them to predict how many beans of each kind will be in the scoop. Many students will notice that the sizes of the beans are different, with the lima beans largest and the black beans smallest and correctly reason that each scoop or "sample" will contain more black and fewer lima beans. Others may raise questions about the "standardization" of a scoop, reasoning that although rules can be made about how level the scoop should be, variations will occur.

Have them each sort their beans by kind and display a chart of their sample. Then collect all the samples so

Table 1
Summary of the Data

Type of bean	Range	Mode	Median	Mean
Lima	8	5	5	5.32
Pinto	7	7, 9, 10	9	8.36
Kidney	7	7	8	8.16
Black	9	7, 9, 10	9	9.36

that the distribution can be further analyzed to predict the actual contents of the entire container, the "population." Help the students to focus on the distribution of the beans by asking such questions as, "Which kind of bean do you think there will be most of?" "Why?"

Establishing a recording system

Ask each student to make a table showing the number of each kind of bean and the total number of beans in a scoop (fig. 3). Show them how to extend the information in the individual tables by comparing the number of any kind of bean to the total to obtain the fractional part. Then encourage them to use their calculators to express that ratio as a decimal. Finally, extend this idea by helping them express each ratio as a percentage. Discuss the data: for example, in Andrea's sample 5 parts of the 28 are lima beans. If 100 beans were in the sample, Andrea would expect to have about 18 lima beans—0.18, or 18 percent. Since we would expect about 32 of 100 beans in the container to be black beans, we might predict that the probability of obtaining a black bean when taking 1 bean from this collection is 32 percent.

Collect the numerical distributions for each kind of bean from the students' samples and display the data on a graph (fig. 4). Show the students how to derive some informal descriptive statistics from their data. For example, the range, or difference between the highest and lowest scores; the mode, or most frequent score; the median, or middle score; and the mean, or average.

Reflecting on the experience

Have the students summarize the data (table 1). Ask such questions as these: Why are three numbers listed as the mode for both pinto and black beans? How do the median and mean of the lima beans compare? Where is the greatest difference between the median and the mean? Which kind of bean seems to show the greatest variation in the samples? With the graphs and summarizing data, ask the students to describe the mixture of beans from the profiles using the terms *range*, *mode*, *median*, and *mean*.

Generating new experiences

Guide the students to analyze the data further. Ask such questions as the following: In one scoop of beans, it is *possible* to get more lima than black beans? Is this occurrence *likely* to happen? How many of each bean would you expect to have if you had a total of 200 beans? 1000 beans? 10 000 beans? This activity offers an application of the concepts of ratio and proportion.

As an extension, encourage the students to plan mixtures of beans with varying percentages of different beans and to collect and analyze samples from those mixtures. Can they predict which sample belongs to which mixture?

Final Remarks

Basic concepts of probability and statistics develop gradually as a result of many experiences. The activities suggested here demonstrate some ways to use readily available manipulative materials to organize such experiences. The four-step approach used in these activities can help the teacher take advantage of the power of manipulative materials in developing mathematical concepts.

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INTRODUCTORY ACTIVITY FOR PROBABILITY SPINNERS

Activity 122: Probability: Using a Spinner

4 5 6 7 8

MATERIALS: Four and Five-Sector Spinners, Master 30.

Have children label a four-column tally chart on Master 30 "R", "G", "Y", "B". Children spin the spinner and record the outcome by placing a tally mark in the appropriate column. Continue the activity for 60 rounds. Total the outcomes and discuss the implications as in Activity 121. The probability of the spinner pointing to any of the four colors is equally likely, i.e. 1 out of 4. Compare these results to the results obtained from flipping a coin.

R	G	Y	B
IIII	III	~~IIII~~	IIII

VARIATION: Repeat the activity using the five-sector spinner labeled 1, 2, 3, 4, 5 and the five-column tally chart similarly labeled. (The probability of the spinner pointing to any of the five numbers is equaly likely, i.e. 1 out of 5.)

Activity 123: Probability: Using a Nonequal Spinner

4 5 6 7 8

MATERIALS: Five-sector spinner, Master 30.

On the white rim surrounding the five-sector spinner, label two adjacent sectors blue. Mark the three remaining sectors red, green and yellow. Discuss the "new" spinner with the children, asking how many colors appear. Have them theorize about the possible outcomes of using this spinner. Have children label the four-column tally charts on Master 30, B, R, G, Y and spin the spinner. Record the outcome by placing a tally mark in the appropriate column. Continue the activity for 40 rounds, total the outcomes and discuss. The probability of the spinner pointing to blue is 2 out of 5, while for red or yellow it is 1 out of 5. (There is twice the chance for the spinner to point to blue as to either of the other colors.) Discuss whether the probability of getting blue would change if the two blue sectors were not adjacent. Relabel the spinner and repeat the activity. Discuss why there is no difference.

name

MANIPULATIVE ACTIVITY

WORKSHEET **24**

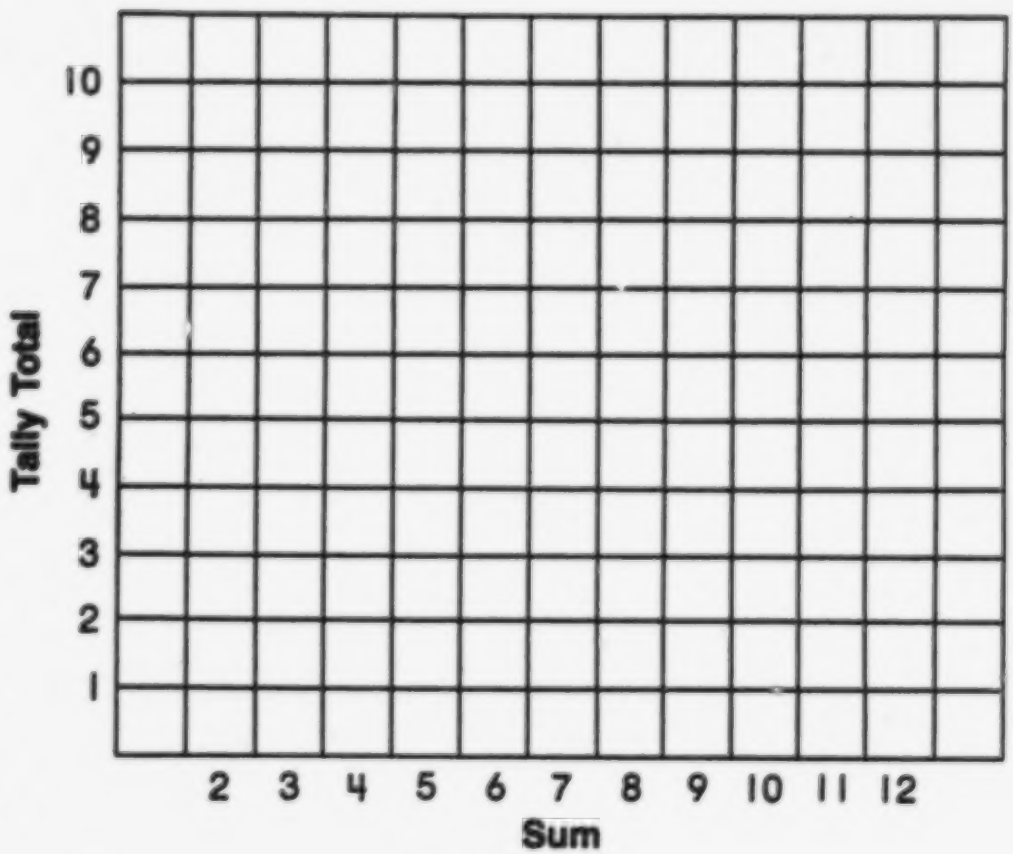
Graphs • Roll the Cubes

Roll a pair of number cubes 36 times. Add the numbers that face up. Show your results in the table below. Make a bar graph.

Answers will vary.

The sum is 8.

Sum	2	3	4	5	6	7	8	9	10	11	12
tally											
total											



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SUGGESTED LESSON PLAN

WORKSHEET **24**

Student Objectives

To perform a number cube experiment.
To record data and construct a bar graph.

Materials

number cubes, crayons

Directions

Demonstrate how to record data using tally marks. Have each child roll a pair of number cubes 36 times, indicating the sum of the cubes by marking a tally mark in the appropriate box on the table. Then have them count their tallies and record the total for each sum in the row labeled *total*.

Help the children to use their data to construct a bar graph. For each sum, the children must find their total on the vertical axis and draw a bar extending to that point.

Answer Key

Answers will vary.

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name

MANIPULATIVE ACTIVITY WORKSHEET 53

Fractions • Coin Toss

Toss a penny and a nickel 20 times. Record each outcome in the table below. **Answers will vary.**

Outcome	Tally	Number for Each Outcome	Fraction of Total Outcomes
			____ out of 20 or ____ / 20
			____ out of 20 or ____ / 20
			____ out of 20 or ____ / 20
			____ out of 20 or ____ / 20

1. What is the total of all the tallies?

   20

2. What is the sum of all the fractions?

   1

3. Your experiment shows that your chance of tossing two heads is about

   Answers will vary. out of 20.

4. Your experiment shows that your chance of tossing 1 head and 1 tail is about

   Answers will vary. out of 20.

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SUGGESTED LESSON PLAN

WORKSHEET **53**

Student Objectives

To perform a probability experiment involving coins.
To record data.

Materials

play money (nickels and pennies only)

Directions

Tell the children that when a single coin is tossed, there are two possible outcomes, or ways that the coin can land. One outcome is heads (H); the other is tails (T). When two coins are tossed, there are four possible outcomes: heads, heads; heads, tails; tails, heads; and tails, tails. These are the outcomes illustrated on the left side of the table on the worksheet.

Tell the children that the probability for each outcome is $\frac{1}{4}$, or $\frac{5}{20}$, since this experiment involves 20 trials. Have them keep this probability in mind as they complete the experiment and answer the questions.

Answer Key

1. 20 2. 1 3.–4. Answers will vary.

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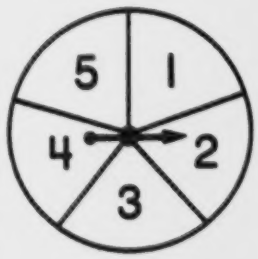
name

MANIPULATIVE ACTIVITY WORKSHEET 54

Fractions • Spinner Fractions

Spin the pointer 30 times. Write your answers in the table shown below. **Answers will vary.**



Outcome	Tally	Number for Each Outcome	Fraction of Total
1			$\frac{\quad}{30}$
2			$\frac{\quad}{30}$
3			$\frac{\quad}{30}$
4			$\frac{\quad}{30}$
5			$\frac{\quad}{30}$

Now solve these problems.

1. How many outcomes are there?

   5

2. How many ways can you spin an odd number?

   3

3. How many ways can the pointer land on an even number?

   2

4. The *probability* of spinning a 5 was 6 out of 30, or $\frac{6}{30}$. Look at your table. Was 5 your outcome 6 times? **Answers will vary.**
   Check your classmates' results.

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SUGGESTED LESSON PLAN

WORKSHEET **54**

Student Objectives

To perform a probability experiment involving a number spinner.
To record data.

Materials

number spinners

Directions

Have children spin the pointer 30 times and record their results in the table. Have them work independently to solve the problems. Discuss their answers to problem 4. Did anyone achieve the predicted probability?

Answer Key

1. 5 2. 3 3. 2 4. Answers will vary.

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Event

37

Statistics and Probability

★

Probability is the study of how often something might happen. The best way to learn about probability is by experimenting first-hand. In this activity, you experiment with spinners.

Name .....

What total do you think you'll get most often?_____

Totals	Number of times
2	
3	
4	
5	
6	

What happened?_____

_____

_____

_____

_____

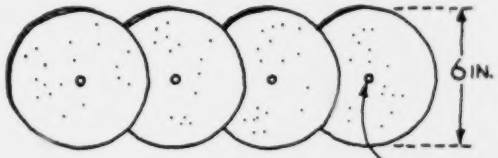
Make another spinner. Now use all 3 spinners and try the experiment. What happens?

Change the numbers on the spinners and try the experiment again. What happens?

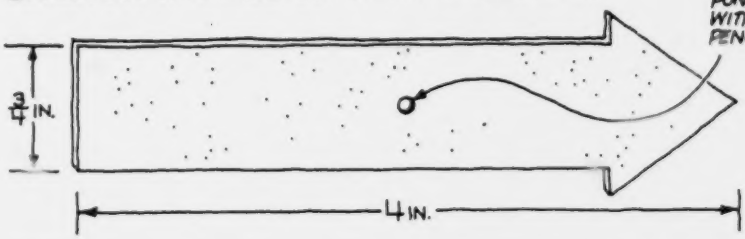
SPINNERS

To make a spinner, you need a paper clip bent like this,

3 or 4 tagboard circles, 6 inches in diameter,

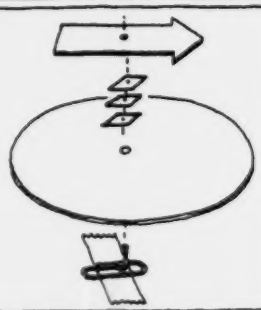


and an arrow this size cut from cardboard.



Cut 3 small tagboard squares. Use them as washers. Spinners really won't spin without them.

To change the face of your spinner, remove the arrow and drop a new circle over the old one.



To be used for Event 37, 43, 49.

Accident Statistics

The following table lists the total number of accidental deaths in the United States in 1979 and the number of fatal motor-vehicle accidents.

Accidental Deaths in the United States, 1979		
	All Types	Motor Vehicle
All ages	105,312	53,524
Under 5	4,429	1,461
5 to 14	5,689	2,952
15 to 24	26,574	19,369
25 to 44	26,097	15,658
45 to 64	18,346	8,162
65 to 74	9,013	3,171
75 and over	15,164	2,751
Male	74,403	39,309
Female	30,909	14,215

Source: *The World Almanac & Book of Facts*, 1979 and 1984 edition, copyright © Newspaper Enterprise Association, Inc. 1978 and 1983, New York, NY 10166.

You are told that a certain person recently died in an accident. Estimate the probability that:

1. It was a motor-vehicle accident.
2. The person was male.
3. It was a motor-vehicle accident, assuming that the person was a male.
4. It was a motor-vehicle accident, assuming that the person was between 15 and 24 years of age.
5. It was a female who was involved in an accident that was not a motor-vehicle accident.

DALE SEYMOUR PUBLICATIONS

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Exploring Probability

Claire M. Newman
Thomas E. Obremski
Richard L. Scheaffer

STUDENT VERBALIZATION

VERBALIZATION

Research Findings:

Putting words to one's thinking often brings that thinking to the conscious level

- Students who were asked to verbalize what strategies they had used to solve a set of problems were more successful on a subsequent set of related problems than students who had not been asked to put their strategies into words.

Questions to Consider:

Do I have students write on the board, come to the board and verbalize the strategies for solutions to the homework and to the class problems?

Do I have students respond in full sentence answers so that they are reinforcing not only what is done to solve a problem but also when the method is to be applied?

Do I have students write on the board, come to the board and explain strategies used in solving test questions?

Why Do Your Students Need To Talk About Mathematics?

Student verbalization allows students to become active participants in the process of teaching and learning. Research has shown that expressing one's thinking through verbalization brings thinking to the conscious level.

Students who take a verbally active part in each lesson through discussing, questioning, explaining and justifying their solutions have allowed their thoughts to be formulated and polished through the additional sense of hearing.

Students who are asked to verbalize what strategies they have used to solve a set of problems are more successful on a subsequent set of related problems than students who have not been asked to put their strategies into words.

It is essential that teachers totally involve students in all aspects of the learning process. The teacher must have students respond during class in full sentence answers so that they are reinforcing not only what is done to solve a problem but also when and how the method is to be applied. The teacher must have students come to the board, write on the board and verbalize the strategies for solutions to the homework, class problems and teacher-made tests. Mathematics instruction becomes most effective when students actively discuss what they are learning.

Through verbalization, students see mathematics as a close knit system of understandable ideas, principles and processes. Students are more likely to retain mathematical learning over time if meaningful discussion activities comprise more than half of their mathematics instruction. Teachers must aim for a regular and consistent involvement of students through verbalization in order to assure meaningful instruction in mathematics. The crucial relationship between student verbalization and understanding is essential to insure student retention.

QUESTIONING

Research Findings:

Effective questioning skills have been linked with pupil's achievement in mathematics. Effective teachers of mathematics ask more questions than do ineffective teachers. They ask more recall questions, and they also ask many more higher-level, process-type questions, calling for explanations rather than only recall. Moreover, effective teachers ask more follow-up questions after a correct answer is given, to probe pupils understanding and help them build concepts. Effective teachers encourage pupils to ask questions. This practice leads students to continually evaluate their own understanding. More effective teachers provide more wait-time for answers to a question.

Putting the values on quick correct responses does not help to establish an environment in which the process of mathematical thinking is encouraged. Questioning is an important part of the teacher's ability to establish a classroom atmosphere conducive to the development of mathematical thinking.

QUESTIONS TO CONSIDER

Does the teacher prepare questions in advance?

Why should the teacher ask more process questions (calling for explanations) than recall questions?

For example:
Do teachers phrase questions to include such terms as "Explain..., Describe..., How..., Illustrate..., Evaluate..., Why..., Compare..., Contrast..., etc.

Why should the teacher have students respond in complete sentences?

How does the teacher ask follow-up questions after a correct response is given?

For example:
What evidence do you have for thinking that?
What method did you use for that problem?

When does the teacher ask for the answer only to a problem and not the process needed to obtain the answer?

What topics should the teacher give or ask for answers about:only homework or test questions? If not what other sources are there?

When are students asked to judge the reasonableness of their thinking and to defend their solutions?

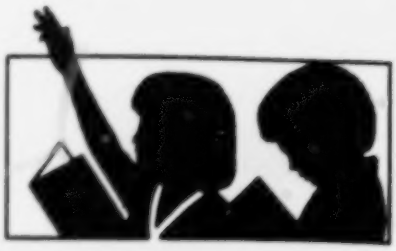
Why does the teacher ask students to listen to each other so that they are trained to judge or comment on each other's responses?

Why should teachers ask students to make predictions, formulate generalizations, consider how ideas can be expanded or shifted, look for alternative approaches, and search for new areas to investigate?

TEACHING

How to Ask Effective Questions

By **Dorothy McCullough** *and* **Edye Findley**



Do you—

- prepare questions in advance?
- adapt questions to various skill levels?
- word questions so as to involve all of your students?
- encourage individuals to take time to answer?
- question students in a way that causes each of them to answer?
- use questions to get feedback on students' learning?

These are vital characteristics of good questioning.

Plan relevant questions

The essence of good questioning is in planning questions that are directly related to the concept or skill being taught. First you need to break down the skill or concept into the fundamental concepts necessary to accomplish the task. Second, you need to analyze the process the student will encounter, which means determining at each step of the task what the student needs to know.

For example, what questions do you need to ask to be assured that a student can solve the following equation?

$$1/3 + \_\_\_\_ = 1$$

Questions like the following indicate that you have considered all components involved in this operation and now your students are prepared to find the missing addend.

1. What does the equal symbol mean? (Expect answers to be expressed in several ways—such as "the same on each side," "an equivalent value," "the left member and the right member represent the same numerical value.")
2. How many thirds are in a whole?
3. What is the value of 1 in terms of thirds?
4. Will the missing number be more or less than 1?
5. Will the number be a fraction? Why?

This kind of questioning does more than prepare a student for success. With careful questioning, concepts and skills are also retaught or reinforced.

Make questions clear and concise

Children have a better opportunity to answer a question when it is concise and clearly stated. Clear statements of questions require careful thought on the teacher's part. Writing questions in advance is a good strategy for keeping questions clear and concise. If you include sample questions in your lesson plans, your questions will be more effective.

Attention should be given to the wording of the question. For example, in the question, "Starting tomorrow and counting Halloween Day, how many days is it until Halloween?" the student knows where to begin and end, and that counting is the process. If you see that children are not answering or are misunderstanding your questions, reexamine the wording. Be flexible—either edit the question or lay it to rest.

Vary the level of difficulty

In preparing questions, it is important to include questions on both concrete and abstract levels. In this way, you involve all learners in moving toward more critical thinking. As you direct questions to students, you need to be aware of the individual student's experience with the topic. When students are appropriately challenged, they feel good about the experience and their self-concept grows. For example, "How many more do you need to have four in a row, Sally?" and "Is it possible for you to accomplish this in one move, Jean?" are two examples that illustrate the use of a concrete question directed to Sally and a more abstract question for Jean. Furthermore, asking questions that cause students to restate the major points of the lesson in their own words helps them to be ready to apply the concept being taught.

Allow adequate wait time

It is phenomenal what a "wait time" of three seconds or more following a question can do to strengthen student response to a question. Both the quality and quantity of the responses are improved.

Rowe (1978) has shown that with a wait time of three seconds or more,

Dorothy McCullough is principal of, and Edye Findley is a first-grade teacher at the Ellisville Elementary School, St. Louis County, Missouri. *This article was developed as a result of a program, "Developing Critical Thinking Skills through Mathematical Games." A key factor in the program was the use of good questioning skills.*

the length of students' responses increases 300–700 percent over the average length of students' responses with a wait time of one second or less. The number of unsolicited but appropriate responses by students also increases. Failures to respond decrease. Speculative thinking increases and contributions by slow learners increase. The time spent in waiting is made up in the responses you gain by making wait time an important element in questioning.

Use both covert and overt strategies

According to Hunter and Russell (1977), teachers should first pose questions to the total group rather than an individual. When this is done, all class members focus on the problem. For example, saying, "I want all of you to think about this question, Why does the answer need to be a fraction?" provokes total group involvement, whereas, "Why does the answer need to be a fraction, John?" involves only one student.

The wording of questions in this manner is important in that it calls for each student to answer the question mentally. As a further example, in a mathematics games situation, the direction, "Each of you be ready to describe your next move. In just a minute I will ask one of you to describe yours." motivates the students to the appropriate level of involvement. The question is then directed to an individual who initiates the discussion.

Remembering to make initial use of covert strategies and follow them with overt strategies produces active involvement and better individual responses. While covert strategies enable more students to participate mentally, the overt strategy is an essential means of observing responses.

Rephrase or redirect questions as needed

Sometimes the most carefully worded question does not produce the results you desire. You may need to supply more information. When this occurs, restate or rephrase the question. Asking the student what information he or she needs to answer the question is an excellent technique.

Breaking the question down into parts can also be beneficial. For example, after asking this question in mathematics puzzles, "Would this piece be the correct choice, Justin?" the teacher could follow up with questions such as "Is that the correct shape?" or "Is that the correct color?" When these questioning strategies have been used and the student is still unable to respond appropriately, involvement of another student is necessary to keep the lesson moving, but such involvement should not be the first step. By involving the second child to obtain additional information, you are assisting the first child in solving the problem at hand.

Summary

For many years, educators have been aware of the vital role of the teacher in establishing discussion skills through questioning in language arts. This same role is just as valuable when it is used to assist students in developing their thinking skills in mathematics.

A teacher's skillful use of questioning strategies provide a model for the students. After demonstrating good questioning techniques, you should plan for a transition that encourages students to participate in questioning and thereby develop a strategy of their own. Through the use of this strategy, students develop their ability to interact with others and, as a result, their thinking skills develop as they solve problems and make decisions. Students become comfortable questioning themselves and others in a variety of situations; they develop strategies of questioning useful in their everyday encounters with mathematics.

Good questioning techniques are a teacher's best tool to insure total class involvement and individual success.

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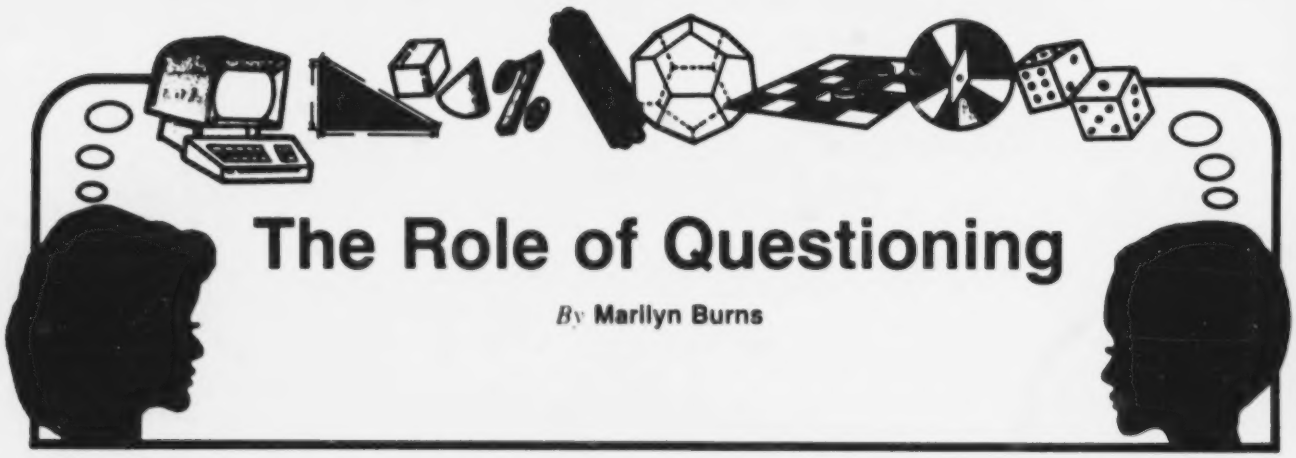

The Role of Questioning

By **Marilyn Burns**

A third-grade teacher was beginning a class lesson in which the children were to work in pairs. When assembling the supplies for each pair of children, the teacher raised the question, "How many pairs of children will we have?" None of the students had a ready answer, nor did they seem to know immediately a way to proceed to get an answer. The teacher asked another question, "What do you need to know to figure this out?" One student suggested that they needed to know how many students there were. The teacher questioned further, "How can we find this out?" The children suggested different ways—having the students count off, counting by rows, or having each student in the class stand or sit down when counted.

After some discussion, when children were engaged in thinking about the problem, the teacher had them work in small groups to figure out how many pairs of students there actually were. When the children's attention was brought back to the total group, the teacher asked for their answers, continuing to prod a bit more: "How did you figure that out? Who did it another way? Maria got another answer; can you convince her that yours is right? If everyone were present today, then how many pairs would we have?"

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That the teacher seized this opportunity for a problem-solving experience is certainly a bonus for the children's mathematical learning. Involving students in real situations is always a plus for mathematics instruction. But more than that, the types of questions the teacher was asking the students to consider were helping to create a classroom atmosphere truly conducive to developing mathematical thinking abilities. Children were asked to consider questions for which answers were not immediately obvious. They were given the opportunity to connect what they did know to what they needed to examine. They were asked to reflect on their thinking. It's just this kind of skill with questioning that is needed to pave the way for teaching children to think mathematically.

Elementary school mathematics instruction has traditionally been geared toward developing arithmetic proficiency. Students spend the bulk of their mathematics instructional time learning arithmetic procedures and practicing them on worksheets or pages of arithmetic exercises in textbooks. Word problems are the usual vehicle for applying these skills, presenting situations for students to translate into arithmetic sentences and then to do the computation needed.

In traditional instruction, the primary goal is to develop computational competence. The emphasis is often on getting right answers, enough right answers to earn good grades or to do well on standardized tests. The teacher or the answer key is the source for revealing to the students the correctness of their answers. And, sadly, it's the quick right answer that is often valued more than the thinking that leads to that answer. What is missing is attention to children's deciding on the reasonableness of their solutions, justifying their procedures, verbalizing their processes, reflecting on their thinking—all those behaviors that contribute to the development of mathematical thinking.

Understanding arithmetic is important for children. However, it is essential that the development of arithmetic understanding be embedded in the context of helping children become powerful mathematical thinkers, able to solve problems, see relationships and patterns, and use numbers confidently to come to decisions. Children need to feel mathematically secure, to enjoy mathematical explorations, to be challenged and stimulated by, and interested in, mathematics. The teacher is the key to making these things happen.

The Teacher's Role

The questions teachers ask deserve careful attention. Students should be encouraged to make conjectures and to examine the validity of their thoughts. They need to search for convincing arguments that support their conjectures or that show discrepancies in others' thinking. They should be flexible enough to consider other approaches and willing to change their position if new light is thrown on the situation. Questions that promote this kind of thinking are of the following types:

What do you think?
Why do you think that?
How come _____'s answer doesn't make sense?
What if _____?
Can you find another way to explain that?
Can you convince me?

The questions will be effective only if the classroom atmosphere supports addressing these kinds of questions openly, even courageously. To create such an atmosphere, teachers need to be models of inquiring learners themselves. Students need to feel that teachers value the exploration of these types of questions as an important aspect of their mathematics learning. Teachers need to emphasize to students that the foremost goal of their mathematics learning is thinking mathematically, not merely getting right answers.

When errors in students' thinking occur, they should not be viewed as unfortunate mishaps but as opportunities for learning. The classroom needs to be a safe place so that new ideas can be tried out—so that children feel free to risk offering their thinking, even if they make mistakes. Persistence with thinking, not speed, needs to be valued. Putting the values on quick, correct responses does not help to establish an environment in which the process of mathematical thinking is encouraged.

Teachers need to ask students to find ways of verifying their solutions for themselves, rather than relying on the teacher or the answer book for verification. Verification may be difficult for teachers to demand; answer keys have long been a part of the teacher's domain in teaching mathematics. In considering this issue, however, we may be helped by taking a careful look at the real-life uses of mathematics. Real problems do not necessarily have one correct solution. At times, several possible solutions exist. Often, an exact solution isn't even required; being close is sufficient. Most important, in real life, it's up to each of us to decide when a solution is "right" or "best." In light of these issues, providing answers does not enhance students' critical thinking skills. On the contrary, giving answers can stop the thinking process. Teachers need to see their major role as focusing students' attention on meaningful questions.

Along with not giving answers, teachers need to ask students to explain their approaches and results, even when they're correct. Often, teachers nod and affirm a child's correct response but question the child only when an error has been made. If teachers question children's responses only when they are wrong, the children soon catch on. The instruction should be structured so that students describe their methods and solutions not only to the teacher but also to their classmates. Time for discussion is needed so that students are encouraged to listen, question, and learn from each other.

Providing answers does not enhance students' critical thinking skills.

A Classroom Example

The horse problem is one I've introduced to many students, from fourth graders to teachers in in-service classes. The problem always produces lively and spirited discussions. More than that, however, this problem illustrates how a teacher can work toward establishing a classroom atmosphere that distinguishes between merely getting a correct answer and focusing on the process of mathematical thinking.

> A man buys a horse for $50 and sells it for $60. He buys the horse back for $70 and then sells it again for $80. Did he earn or lose money, and how much? Or did he come out even?

When presenting this problem, I ask the students to think individually about the problem at first, without discussing it with each other. (If you haven't thought about this problem before, try doing so now.) After collecting students' answers, I have them share their thinking in small groups. Finally, I ask for volunteers to report both their solutions and the thinking that led them to their solutions.

Answers to the problem usually vary, but a core collection is always offered. Some students feel that the man earned $10, others feel that he earned $20, some argue that he lost $40, some feel he wound up $30 ahead, and some think he came out even. At times, students argue that you can't tell because you don't know how much it cost to feed or board the horse; when that argument arises, I ask the student to focus on the situation as it is rather than consider that complication. Interestingly, the same range of answers usually emerges from groups of elementary and secondary school students and teachers.

When you discuss this problem with a class, it's important to remember the goal of fostering the development of mathematical thinking. It's fairly clear to students that somehow they have to add or subtract the $50, $60, $70, and $80 amounts. It's not the arithmetic itself that poses a quandary. It's thinking about how to make sense out of the situation that is the problem. The struggle for meaning needs to be nurtured by questions that encourage the struggle:

Who has a different solution? How did you get it?

- How did you arrive at your answer?
- Did someone get the same answer but by a different way of reasoning?
- Who has a different solution?
- Can you explain a flaw in someone else's reasoning?
- Can you convince someone who disagrees with you?
- What if a man bought a horse on the first transaction and a cow on the second? Would that change the problem?

Very often, after some discussion of possible solutions, a student will ask what the real answer is. It's at this point that additional skill is required

of the teacher. Once again, it's important to remember the goal of promoting thinking. By telling the correct answer, you run the risk of having students stop their thinking. If the answer is the primary goal, students have no further reason to continue with the problem once the answer is revealed. Instead, the teacher needs to support the students' thinking. It's useful to have the students act out the problem, using slips of paper for $10 bills. It is helpful to encourage students to discuss the problem with their friends and families. Keeping mathematical exploration alive is the key.

Encouraging continued thinking may be frustrating for students. You do not want to discourage students so that they stop thinking, nor do you want to leave them with misconceptions. Making the decision not to provide an answer, but instead to keep the problem alive, is one of those judgments that teachers continually have to make. When doing so, you need to discuss your action with the students, creating an awareness of the existence and importance of mathematical thinking. This awareness needs to be nurtured consciously and continually by talking about it whenever the opportunity arises.

Final Thoughts

Teachers need to make questions of the types described a regular part of children's mathematical experience, so they come to regard these questions as a basic ingredient of their learning of mathematics. It is not enough for children merely to learn the processes of mathematics. They need to increase their understanding of what mathematics is and learn to use those processes.

It's important to remember that developing the ability to think mathematically is not a lesson objective but a long-range goal. Time is often a concern for teachers—you've got just a year to get through an enormous amount of material. Natural learning, however, doesn't happen on a schedule and often requires more time than schools are organized to provide. Discussions with questioning students take time. It's essential that teachers give the time needed for children not only to work through activities that promote thinking but also to reflect on that thinking whenever possible.

Teachers need to hold the vision of positive, open, inquiring children, who willingly explore mathematics with enthusiasm. Children's experiences in mathematics need to be exciting and fun, not in the sense of easy or trivial but in the sense of engaging them fully and keeping them involved in the investigation.

Mathematics involves a sense of investigation.

Classroom experiences must extend beyond the goal of arriving at correct answers. Children must be asked to judge the reasonableness of their thinking, to defend their solutions. Further, children's classroom experiences need to lead them to make predictions, formulate generalizations, justify their thinking, consider how ideas can be expanded or shifted, look for alternate approaches, and search for—even seek—those insights that, rather than converging toward an answer, open up new areas to investigate. Children are capable of this quality of work in school mathematics. To expect less is pointless and demeaning. Questioning is an important part of the teacher's ability to establish a classroom atmosphere conducive to the development of mathematical thinking.

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The Art of Questioning

Effective questioning is the tool with which teachers develop logical thought, stimulate the imagination, bridge the gap between previous learning and new learning and give direction to the learning process. When skillfully phrased and strategically used, questions become one of the teacher's most flexible instructional tools. On the other hand, poor questions and poor questioning technique interfere with learning by creating confusion, discouraging pupil involvement and inhibiting the sequential development of concepts.

General characteristics of effective questions:

- They are kept short.
- They are simply phrased and direct in language.
- They are logically sequenced, the response to one question leading into the next, with each question calling for only one or two points in the chain of development.
- They are designed to require pupils to think, rather than respond instantly with an association. Whenever possible, they elicit a sustained response rather than a one- or two-word answer.
- They are frequently personalized to grasp the pupils' attention: "How would you explain to a classmate who was absent yesterday our method for changing common fractions to decimal fractions?"

Twelve Types of Questions to Avoid: A Dirty Dozen

Teachers sometimes develop ineffective questioning habits that are difficult to break unless conscious examination in precisely what kinds of questions they have been asking takes place. Ineffective questions come in many varieties, and are ineffective for various reasons. We have all heard them, and have perhaps used some of them. Here is a brief compilation of some of them:

(1) The Multiple or Overlaid Question:

Example: What is the number above the fraction bar called, and what is the number below the fraction bar called? How do we reduce fractions?

Criticism: By asking for more than one response, the teacher is limiting participation to only those pupils who know all of them. Many more pupils might be able to respond if a single question were asked at a time.

Suggestion: Ask the three questions separately, waiting until one question is answered before going on to the next.

(2) The Vague or Ambiguous Question:

Example: How about rectangles?

Criticism: The question doesn't direct pupil thinking in any particular direction. A multitude of irrelevant responses can be made.

Suggestion: What kind of angles do the sides of a rectangle form?

(3) The Choral Question:

Example: Were we right to multiply by 100? (Y-e-e-e-e-s!)

Criticism: Pupils are not made to feel personally responsible for the accuracy of their response. They are encouraged to hide behind the anonymity provided by choral response. Different responses to the same question are uttered together, creating confusion for individual pupils who may not know which one is being accepted as correct.

Suggestion: Do you think we were right to multiply by 100, ... Joseph?

(4) The Leading Question:

Example: There are 6 fours in 24, aren't there?

Criticism: A pupil who is not sure will promptly agree with the teacher without thinking about his response. He has been invited merely to nod his head in acquiescence.

Suggestion: How many fours are there in 24,... Sherry?

(5) The Tugging Question:

Example: The area of a rectangle is found by multiplying the length by the, ____...? W____, w____. Come on now; it starts with "W."

Criticism: The question reduces the discussion to a guessing game, often not even based on the subject matter but on the recall of words by virtue of their sounds.

Suggestion: How is the area of a rectangle found? OR: What two measures do we multiply to get the area of a rectangle.

(6) The Teacher-Oriented Question:

Example: Who can tell me how many places to move the decimal point when multiplying by 1000?

Criticism: The question contributes to a feeling that the class is organized around the teacher, and that what is most important is satisfying the teacher's requirements. It does not contribute to a group feeling of a class working together to learn.

Suggestion: Who can tell us how many places to move the decimal point when multiplying by 1000?

(7) The Whip-Lash Question:

Example: The unit of measure that is equal to 1000 meters is called the what?

Criticism: The question is not perceived as a question by the pupils since it does not begin with an interrogative. It sounds as though the teacher is making a declarative statement. Only at the last word do pupils learn that they are being asked for a response, and thus have no time in which to think about it.

Suggestion: Include one of the words: how, what, why, when, etc. near the beginning of every question. "What is the name we give to the unit of measure that is equal to 1,000 meters?"

(8) The Yes-No Question:

Example: Is the radius of a circle twice the diameter?

Criticism: Every pupil has a 50% chance of answering correctly by merely guessing "yes" or "no." The question does not encourage thought, nor does it encourage pupils to use the vocabulary that they have learned.

Suggestion: By what number would you multiply the measure of the radius to find the measure of the diameter?

(9) The Echo Question:

Example: There are four quarters in a dollar. How many quarters are there in a dollar?

Criticism: The question calls for only the most rudimentary form of response: parroting. It does not stimulate thought, and a pupil can answer it correctly without really understanding the concept.

Suggestion: Wait for some time to pass before asking pupils to recall a previously developed fact.

(10) The Question Addressed to a Specific Pupil:

Example: Robert, can you move the hands on this clock to make it read one-thirty?

Criticism: By starting with the name of a specific pupil, the teacher has turned the discussion into a one-to-one exchange. Other pupils are not encouraged to think about their responses, since the question appears to have been designed for only a particular pupil.

Suggestion: Ask the question, pause to give all pupils time to consider their responses, and then call on a particular pupil. Who can move the hands on this clock to make it read one-thirty? ... Robert?

(11) The Guessing Question:

Example: Is 25% equal to one-half or one-quarter?

Criticism: Like the Yes-No question, this type of question limits thinking and creates a situation in which any pupil, regardless of his understanding of a concept, has a 50-50 chance of answering correctly.

Suggestion: What common fraction is 25% equal to? OR: Is 25% equal to one-half or one-quarter, and explain why you made your choice.

(12) The Re-phrased Question:

Example: What name do we give a half-circle? I mean, when you draw a diameter, it divides the circle into two parts. What do we call each part? When a circle is divided into two halves, what do we call each half?

Criticism: Pupils are still dealing with the first version of the question when they are presented with second and third versions. While listening to these re-phrasings of the original question, thought is diverted and clarity is inhibited.

Suggestion: Ask one good question clearly expressed and remain silent while pupils are considering their responses to it. For example:
If a diameter is drawn in a circle, it divides the circle into two equal parts. What name is given to each half-circle?
Pupil Responses

Related to the art of effective questioning is the skill with which the teacher reacts to, or accepts pupil responses. Pupils can be encouraged to formulate their responses in such a way as to make them of value to the entire class. Teachers can further encourage pupil involvement and enthusiasm in the learning process by the ways in which they use pupil responses. Some general guidelines follow:

- Encouraging pupils to comment on each other's responses involves pupils in the learning process to a greater extent than reserving the right to comment as an exclusive right for the teacher: Do you agree with Mario's explanation? How would you correct Mary's answer? What is another way of solving Renee's problem?

- Following up an incorrect response with a sequence of questions designed to help the pupil correct his own answer is much more encouraging than merely skipping over it.

- Appropriately underscoring pivotal responses is helpful to pupils by pointing up those key concepts which are at the heart of the lesson. Recording them in the form of a summary at the board, commenting on their importance verbally, asking other pupils to re-phrase them and paraphrasing them yourself are all ways of accomplishing this.

- Refraining from repeating pupils' responses encourages pupils to address their responses to the class as a whole, rather than to one individual, the teacher. When the teacher becomes someone on whom the class can rely for the repetition of all pupil responses, it is no longer necessary for pupils to listen carefully to each other.

Interaction between pupils is minimized, and the atmosphere of a teacher-centered classroom is fostered.

- Resisting the temptation to be immediately responsive to pupils' needs by answering their questions directly encourages wider pupil involvement in the lesson. As much as possible, re-directing individual pupils' questions for consideration by the whole class is to be preferred over a one-to-one conversation between that pupil and the teacher. There are very likely other pupils in the room who can benefit from hearing the question, thinking about their own response to it, and hearing the answer.

- Reacting to correct pupil responses in a variety of ways is more encouraging and rewarding for pupils than acting in a consistent, habitual way, such as always saying "Good."

- Remaining flexible with regard to what constitutes a correct response acknowledges pupils' contributions to the lesson more sincerely than rejecting all responses other than the particular one that the teacher had in mind originally.

- Encouraging pupils to ask as well as answer questions involves them even more deeply in the learning process. But this entails more than merely asking, "Are there any questions?" Sometimes, individual pupils have to be drawn out when it is obvious that something is bothering them but their question is not forthcoming. When pupils do ask questions, it is important that they not be made to feel embarrassed or ridiculed for having asked it. The question should be given respectful consideration by the whole class, or, at least, answered sincerely in private. The pupil asking the question should not be made the subject of a cross-examination, the surest way of guaranteeing that he will never ask another.

QUESTIONING IN THE MATHEMATICS CLASSROOM

You will find below some questions which might be asked by teachers of mathematics. Some of them are well constructed, in good form and in good English. Others are faulty in one way or another. Criticize each question according to the following pattern:

a. If you think the question is a good one, indicate this fact, and explain why you think it is good.

b. (1) If you think the question is a poor one, indicate this fact, and explain why you think it is poor, and

(2) rephrase the question in a way which will make it satisfactory.

1. Anthony, how do we round 38,464 to the nearest thousand?
2. How do we find the perimeter of a rectangle?
3. The product of 36 and 48 is what?
4. Is 1 a prime number?
5. What is the difference between congruent figures and similar figures?
6. Is 5 a prime number or a composite number?
7. We can change a fraction with a denominator of 100 to a decimal, how?
8. The difference between a chord and a radius is what?
9. How is a parallelogram different from a rectangle and how is a parallelogram the same as a rectangle?
10. Ricardo, how do we find the perimeter of a rectangle?
11. In what situations are negative and positive numbers used?
12. In how many ways can we arrange 3 different books on a shelf?

QUESTIONS WHICH ENCOURAGE QUESTIONS FROM, AND INTERACTIONS AMONG, STUDENTS

-Give me another example.

-Do you believe that?

-Will that work with (fractions), too?

-How do you know that?

-Are you sure? Let's test it.

-Why do you and he disagree?

-Can you convince the majority that they're wrong?

-That seems to work. Will it always?

-What do you mean by that?

-Say it another way.

-How can we simplify this?

THAT ART OF QUESTIONING/GETTING BETTER STUDENT RESPONSES

1. Pause after asking a question.
2. Avoid questions which require only a yes or no answer.
3. Avoid answering your own questions.
4. Follow up student responses with the question "why?"
5. Limit the use of questions which rely almost completely on memory.
6. Avoid directing a question to a student for disciplinary reasons.
7. Avoid repeating a student's answer.
8. Follow up a student's response by fielding it to the class or to another student for a reaction.
9. Avoid giveaway facial expressions to student responses.
10. Avoid asking questions that contain the answer.
11. Avoid calling on a particular student before asking the question.
12. Avoid calling on a particular student immediately after asking a question.
13. Ask questions that are open-ended.
14. Avoid labeling the degree of difficulty of a question.

USE OF QUESTIONS

-to motivate

-to challenge (Do you believe that? What do you mean by that?)

-to provoke student interaction

-to get students to evaluate (Are you sure? How do you know that?)

-to focus on process (What method did you use?)

-to guide

-to diagnose

-to review

-to encourage exploration

-to invite student question (What questions does this information leave unanswered?)

-to enhance transfer (How could you use that result in this new situation?)

IMPROVING STUDENTS' THINKING SKILLS IN MATHEMATICS CLASSES THROUGH QUESTIONING

Skillful questions serves:

- to direct activity
- to promote discussion which will extend knowledge and awareness
- to stimulate thinking
- to measure accomplishment by testing information and understanding
- to promote student interaction

Devising Thought Provoking Questions:

When interested in developing ideas, concepts, insights, and interrelationships, teachers should use questions that ask: how, why, illustrate, show, summarize, compare, contrast, explain, describe, evaluate, etc.

Questions that promote students' thinking skills:

Explain the ...

What do you think that?

How do you think that?

How can _____'s answer be corrected?

What if ...?

Can you find another way to explain that?

Can you convince me?

What reasons or evidence ...?

How are ...similar? How do they differ?

Compare ...?

Where does _____lead you?

Why did you ...?

How did you determine that the answer is correct?

What's wrong with the statement ...?

Correcting Incorrect Statements:

By appropriate questioning the teacher can correct partially or fully erroneous statements. For example, the teacher may ask the entire class:

- With which parts of John's answer would you disagree? Why?
- What facts have we learned that tend to contradict Bill's answer?
- What's wrong with the answer just given?
- How might we revise Jill's answer to make it more accurate?

Remember, the task of the teacher is to give credit for the part of the answer that is correct and attempt to have the incorrect part of the answer improved by questioning. e.g., How could you improve the answer just given?

Questions from Students:

In order to distribute the responsibility for learning and make the class share the process, the teacher should refer students' questions to the entire class by asking: Who can help Donald by responding to his question?

If you think math is all numbers, read how one teacher uses his students' writing as a quick, sure gauge of how well they're doing and where they need help.

WRITING LETTERS To LEARN MATH

BY BILL KENNEDY

Have you ever wondered, as I have, if students sometimes think of mathematics as a game of "fetch"? It works like this: You toss out a question. The students plunge into the tangled thicket of numbers; they quickly return with what they think is the right number and sit like spaniels—tails wagging, prey clenched between their teeth—waiting for your approval. If you say no, they rush off after another likely looking number.

You want more than just fetch-and-carry answers from students. I do too. I want them to be able to decide by themselves whether an answer is plausible. And I've developed a way to move them toward that goal. I call it "writing in math." It's an integral part of my math curriculum, and it involves having the students write letters, learning logs, word problems, and approaches to solutions.

Writing Letters

When we're about halfway through a math unit, I'll stop 10 minutes before the end of class and ask the students to write me a letter about what we're studying. The letter has to include three points: (1) what they understand (with examples) about the work we're doing in class; (2) what they don't understand (with examples); and (3) what they're wondering about (specific questions). As with any assignment that's to be collected, I limit the letter writing to just one class on a particular day—the best way to stifle my interest is to drown myself in six sets of papers at once.

Writing these letters allows students to confess their ignorance of things they'd never admit to in class. Even self-assured, bright, sociable eighth graders are reluctant to say publicly that they've forgotten how to divide fractions. But in the privacy of a letter to their teacher, they can afford the risk of saying "I don't get the part about...."

That night, I read the letters; it doesn't take much time because I'm not correcting them, just looking for information. Then I take notes for preparing the next day's class. My plan depends on what the letters reveal. I may meet with students individually or—if discretion allows—with small groups to discuss their "I don't get the part about..." statements. Or I may review something with the whole class.

In the short time it takes me to read the letters, I learn more about my students' needs than I could in two or three class periods of observing and asking questions, more even than from a test that would take me hours to prepare and grade. In class the next day, I can sometimes clear up their questions with a brief explanation. Other times, the letters show me that I've gone too fast or taken too many leaps, so the next day I back up and retrace my steps.

Keeping Logs

Besides writing letters, my students also keep learning logs. In these logs, they maintain a regular record (daily, weekly, or at the end of a unit) of personal learning as it happens. The key feature of these logs is that the students write in their own language.

Essentially, the students are writing to themselves about what they're learning. Here's what one girl wrote in her log about square root: "Square root is like a question that asks, what number times itself will be the number under the square root sign (which is called a radical)? One thing to remember is that if a number has

an odd number of zeros, it is not a perfect square."

Note the analogy and the reminder to herself; both are good memory devices. Note too that she's *making* notes, not taking notes. This distinction is central to the use of a learning log. Taking notes is copying someone else's information; *making* notes is writing interpretative comments and personal reminders such as "Ask about this" or just "Why?"

Devising Problems

Frequently, *asking* a good question is more difficult than answering one. That's why, from time to time, I have my students demonstrate their understanding of mathematical concepts by writing word problems. Sometimes I have them supply the data (for example, "Write a problem involving the use of percent"); other times I supply the data (for example, "Write a problem using the numbers 200, 400, and 600").

As a variation, I might ask them to write a problem that has too little data to be solved. Once when I did, a student responded with the following: "If Tom bought four pencils for 20¢, how much did each pencil cost?" Some students said the problem was too simple; obviously each pencil cost 5¢. But the author pointed out that the problem didn't say all the pencils cost the same amount. Everyone learned a lesson that day about underlying assumptions.

Before I assign problem writing, I show an example on the chalkboard or overhead projector. That way, the students can ask questions ahead of time. Once they've written their problems, the students gather in groups of three or four and take turns presenting them. The exchange within these groups helps the students better understand the process of writing problems. Gaps and omissions become apparent; if students are confused, they immediately demand clarification from the author. Most important, the students learn from each other.

After the students have heard each other's suggestions and revised their problems accordingly, I collect the papers and read them. With the authors' permission, I duplicate some for other classes to solve or to use as models of well-written problems.

Approaching Solutions

Frequently, on Mondays, I'll give my students a problem with the assignment that by Friday they hand in a written report on how they went about trying to solve it. During the week, I take a few minutes of each class period to answer questions about the problem and to review concepts that may be needed for solving it.

I encourage the students to write in simple terms and to express their feelings about the process as they try to solve the problem. Here, for example, is a problem I posed, along with excerpts from how two students approached a solution. (The problem is taken from *Problem of the Week* by Lyle Fisher and William Medigovich, Dale Seymour Publications.) "At some stops a certain bus picks up 5 people. At other stops, it picks up 2 and at the same time lets off 5. There are no other stops but these. It starts its run empty and picks up 5 people. At the end, it has 1[illegible] people aboard. If the number of stops is greater than 17, what is the least number of stops the bus makes?"

From John's report: "I tried just going through the problem a few times without great results. So I then had the idea of trying it backward, starting with the answer and thinking that for the second-to-last stop you need either a 6 or a 14. I continued doing this until I found a way to do it. I made a chart to help me...."

From Rachel's report: "I found out the least number of stops the bus can make by starting with 5 people (obviously). Then I looked at every possible number of people that could be on the bus. (For example, it could be either 2, because 5 minus 3 is 2, or it could be 10, because 5 plus 5 is 10.) I continued this system of breaking down the numbers in all possible ways until I got a lengthened chart...."

I encourage the students to work in groups as they try to solve the problem—though each has to write his or her own final report. By challenging each other's assumptions and checking each other's work, they learn to recognize many plausible approaches. They also learn that in problem solving, creative strategies are more important than quick solutions. The strategies, I tell them, will carry over into life, not the solutions.

Because of m, emphasis on strategies, I'm not dismayed when a student's report doesn't include a correct solution. I believe students learn as much from their failures as from their successes, so keeping a record of successful and unsuccessful attempts helps them understand what works and what doesn't. Even when they have a solution, I want them to ask "Does this make sense?" If the answer is "No," I want them to have a ready, systematic process for finding their way to other solutions. Writing helps provide this.

To emphasize the multiple-approaches method, I read aloud (with their permission, of course) the processes that different students reported in pursuing a solution to the same problem. I also read aloud any report that's a good example of the problem-solving process.

Final Benefits

"Writing in math" can reveal problems, clarify thinking, and generate ideas and questions. If you want to try it with your students, I have a final word of advice. Because this writing is tentative, exploratory and personal, it's also a little scary. To open themselves that much, your students must first know they can trust you. They must know you won't make judgments about their personal worth on the basis of what they've written, and that you'll keep confidential any admissions of ignorance—especially when these refer to topics supposedly learned years ago.

If that trusting atmosphere exists, your students—and you—will find that "writing in math" opens a door to awareness and understanding not just of math, but of how they think and learn. •

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WRITING HELPS RELIEVE MATH ANXIETY

Sometime in mid-October, after we've come to know each other a bit, I ask my students to write about their overall experience with math in elementary school. I tell them to be specific about how they felt about math and to include any humor, even if what seems funny now didn't seem funny when it happened.

The papers frequently have a common thread. A sixth grade girl wrote, "I can't do math. I've never been able to do it. My mom says not to worry—she could never do it either." An eighth grade boy wrote, "I have been in math for 7 years now, and I still get scared on tests. I do fine until the test, then everything seems to pile up and I give up."

These comments, and others like them, remind me that many people label themselves as nonmath types. Some, like the girl, see it as genetic; others accept the label from parents or teachers. All have the same reaction: math anxiety.

For those who approach math as fearfully as they would a coiled, hissing rattlesnake, writing provides an antitoxin, a way of dealing with the poisonous math anxiety. Simply writing about their fears, knowing they will be heard, helps students relax and enables them to keep trying.

Using Writing to Learn Mathematics

By CYNTHIA L. NAHRGANG, Hopkins High School, Minnetonka, MN 55345
BRUCE T. PETERSEN, Michigan Technological University, Houghton, MI 49931

Writing has a place in a mathematics class. It provides a new and interesting mode of communication between the student and teacher, as Watson (1980) reported:

> This two-way conversation has become beneficial to the class. The students realized I hear them and care. They seem to have looked inside themselves and to have seen what they could do to help solve their mathematical problems. Many of their grades improved.

Written assignments in mathematics classes also afford students the opportunity to organize their thoughts and, at the same time, improve their writing skills (Johnson 1983).

The purpose of this article is to show that writing can be used in mathematics to enhance learning. The act of writing gives students the opportunity to formulate, organize, internalize, and evaluate concepts. Emig, who has done extensive investigation of the thinking processes involved in writing, argues that writing "represents a unique mode of learning—not merely valuable, not merely special, but unique" (Emig 1977). The idea that writing is a powerful aid to learning has also found much support in cognitive psychology. For example, Bruner (1966) states, "the more we know about the manner in which it [writing] can aid thought... [the more it] leads me to put language at the center of the stage in considering the nature of intellectual development."

Our experience indicates that the most effective method of using writing to help students learn mathematics is through the use of journals. These in-class writing exercises offer students the opportunity to work informally and personally on mathematical concepts, using their own language and real-world experiences. They can play with a variety of concepts without the fear of the formal evaluation of their writing and their mathematical skills. Our experience in college-level classes, from algebra through calculus, makes us feel confident that these short but powerful writing exercises enhance learning. The journal goes beyond rote learning and challenges the students to use intellectual skills. Myers relates writing to individual learning: "When doing this simple activity [writing], students will be forced to relate information from the lecture to what they already know and to organize and synthesize it so that the concept becomes their own" (Myers 1984).

> Journal-writing sessions were substituted for quizzes.

The Journal

The journal is a diarylike series of writing assignments. Each assignment or entry is a short written response to an instructor's question, statement, or set of instructions. All responses are written in prose rather than in the traditional mathematical style of numbers and equations. The students are directed to use an expressive writing style so that their responses are spontaneous and all their thoughts are recorded. Figure 1 is an example of a journal entry made by a student in technical writing in response to the following assignment: "Describe a real-

During the preparation of this manuscript, Bruce Petersen died. His name appears on this article as a tribute.

$a^{-n} = \frac{1}{a^n}$ is an inverse rule so, if any power is raised to the neg. it becomes a fraction, also making it positive.

You inverse what it says, you switch it around

So, what in life is switched around or inversed (which means the same thing?)

What can I turn around in my life - yet keep it the same; to mean the same? (yet it is non-mathematics)

- relationships
- feelings
- communication - the way I say something or communicate

The english language and communication could easily apply to this rule.

We may change or turn phrases around yet keep their meaning the same.

Fig. 1

life, nonmathematical analogy to the rule $a^{-n} = 1/a^n$."

Journals have two basic functions. First, they allow students to proceed at their own rate and to converge on an understanding of mathematical concepts using their own experiences. Second, they provide teachers a unique diagnostic tool; the writings of students immediately reveal areas of confusion and expose misunderstandings of mathematical concepts.

Journals were introduced into our mathematics classes at Michigan Tech with a minimum of difficulty. These classes were conducted over a ten-week period. Each class included about ten unannounced quizzes that accounted for 20 percent of a student's final grade. By substituting twenty journal-writing sessions for these quizzes, we gave our students two opportunities a week for writing in their journals. However, the flexibility—one of its advantages—of the journal-writing technique permits the instructor to devote extra assignments to more difficult mathematical concepts.

Journals are an excellent diagnostic tool.

Most journal entries require three to seven minutes, depending on the type of assignment. Our experiences have shown that as students become more familiar and comfortable with this technique, their responses become better organized and thus easier to read. In all classes, we found that students' experiences in writing in journals led to definite improvements in their ability to organize their responses.

The method of evaluating journals became an important question. Initially, grading each journal seemed to be most appropriate, with the average of all journal grades being a part of the final grade for the course. We immediately realized that this method restricted the potential of this learning tool. By assigning grades, we were indirectly telling the students how they should process concepts, which was defeating one of the important attributes of journal writing—individualized learning and discovery. However, students seemed to deserve some type of credit for their efforts. After considering several alternatives, we decided to give one or two points toward the next major examination on the basis of the students' comments about concepts. The most appropriate responses to mathematical concepts were given the maximum extra credit allowable. A lack of effort received no credit. In addition, we often added appropriate comments (e.g., "excellent," "good," "incomplete," or "please revise"). We found that students appreciated the teacher's comments as much as they did the extra credit. Finally, all mathematical errors were corrected. Although these errors were a secondary consideration, they were brought to the students' attention.

An Analysis

Journal entries demonstrate individual learning activities. Various types of assignments were made that coincided generally with the levels specified in Bloom's taxonomy (Bloom, Hastings, and Madaus 1971). During a section on factoring, students were asked to do the following journal assignment: "Discuss the following statement: 'Factoring and finding a product are reverse processes.'" The objective of this particular assignment was to have students collect ideas, internalize the concept of factoring, and examine its significance in finding a product. This task was assigned early in the study of factoring and finding products of polynomials. Observe how each of the following students begins to understand the process of factoring in a personal frame of reference. Student A wrote:

> Factoring and finding a product are reverse processes. They result as reverse processes but the way you go about it isn't reverse because the opposite of multiplication is division. The equations are interchangeable though and can be called. . . .

At first, this response appears to be nonsense. Notice, however, the student's at-

tempt to link multiplication and division to finding a product and factoring. This link is important because it gives this student a bridge to familiar concepts, multiplication and division.

Even in the response of student B, a marginal student, we can see the cognitive value of writing:

Factoring and finding a product are reverse processes. Factoring I guess is to make finding the answer much easier. I really am not sure how much it helps because I can't see much point in it. I guess that it puts the problem in simpler terms.

First, the student restates the assignment and then works toward a basic but correct evaluation. The importance of this student's response can be seen in the last sentence. The student discovers one of the main purposes of factoring, to simplify an expression so that it can be used in further operations.

Student C shows an understanding of the mechanics of the two processes but demonstrates no insight into the concept or purpose of factoring.

I believe that factoring and finding a product are reverse procedures. When you find a product of two binomials that is multiply:

$$(x + 2)(2x - 1) = 2x^2 + 3x - 2$$

When you factor a trinomial you end up with two binomials:

$$2x^2 + 3x - 2 = (x + 2)(x - 1)$$

This response also illustrates a difficulty with evaluating statements. The more equations and mathematics we see in a response, the less we know about students' understanding of a concept. For this reason we always encourage our students to answer with words and sentences rather than with equations.

In the following example, notice how student D collects ideas on finding a product and factoring, then, in the third sentence, attempts to answer self-generated questions, with a satisfactory conclusion.

Combining factors to find a product is an application of algebra that allows factors to be combined into either monomials, polynomials, or trinomials, or whatever. Factoring is the process of taking monomials, polynomials, binomials, trinomials, or whatever and reducing them to the terms that multiply together to form them. Thus, factoring is taking polynomials apart, while finding the product is putting them together. Actually the two are very similar in the processes used. One uses primarily multiplication, the other division.

The student collects, internalizes, and finally evaluates. For the student, the journal is a progression toward the understanding of a concept. For the teacher, the journal is a record of a student's thinking and thus an excellent diagnostic tool.

Given the same assignment, some students focused their thoughts on isolating a common factor. For this method of factoring, students sometimes confused division with the isolation of a common factor. They did not grasp the idea that division is actually contained within the process of isolating a common factor. The success or failure of students in recognizing this difference is clearly demonstrated by the responses of students E and F, respectively:

Factoring is sort of dividing $(3x^2 \div 6x)$ when you factor out the $3x$ you get $3x(x + 2)$. To multiply $3x(x + 2)$ you do the opposite and you get $3x^2 + 6x$ again. So finding a product and factoring are the same simply reversed.

When factoring you remove all common factors from an equation. These are removed usually by dividing them out of the whole equation. Now to get back to the original equation from the factored one, you have to multiply the factored equation by the numbers and variable, etc., that were removed. This shows that factoring and finding the product....

Student E doesn't separate the two ideas as well as student F. In the second sentence student F clearly understands that division is a part of the process of isolating the common factor.

A subsequent assignment in the same class required students to translate the concept of finding a product and factoring to a nonmathematical event. Students responded to the following prompt, "Think of a nonmathematical relationship that is analogous to the process of finding a product and factoring." Student G gave one of the better responses:

One relationship similar to finding a product vs. factoring is that of taking a carburetor apart and then putting it back together. When the carburetor is together it is difficult to clean and repair. Therefore it is disassembled to make it easier to work with. This is analogous to factoring.

When the parts of the carburetor are in working condition, they must be put back together before they will work as desired. This is analogous to finding a product.

This journal assignment differs from the other assignments because students need to compare nonmathematical experiences to what is being learned in class. The response of student G demonstrates an overall understanding of factoring and finding a product by comparing these processes to the disassembly and assembly of a carburetor. He understands their relationship to each other and is able to connect the two processes. His analogy also illustrates that factored and product forms have different mathematical uses. In this assignment the student had to collect information from both mathematical and nonmathematical experiences and then connect them logically. This example shows that in one short journal a student can use a variety of intellectual skills, such as synthesis, interpretation, translation, analysis, and evaluation. We believe that with continued use of these skills, students grow both in the understanding of mathematical concepts and in the ability to express that understanding.

Conclusion

We adapted the technique of journal writing to instruction in mathematics classes. We attempted to promote an improved internalization of mathematical concepts and to see how journal writing could be incorporated in these classes. This study examined students' test scores, attitudes, and writing (the latter by the Writing Apprehension Test) as well as student and teacher evaluations. The data have not established any strong relationships among the various attributes examined. An examination of journals, however, clearly indicates that students used their journals to think about solving problems associated with mathematical concepts. In addition, most of the students' responses to a questionnaire indicated that students viewed journals as a worthwhile inclusion in mathematics classes.

Any conclusions derived from this study must be viewed and extended cautiously. They are specific to a particular isolated setting. The study has led to guarded conclusions and has generated many more questions than it has answered. The investigating team hopes that these efforts are more of a "first step" rather than a "stopping place" and that further study will establish more definitive relationships between writing and mathematics.

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A New Contributor

When the Process of Writing Becomes a Tool for Learning

Wendy F. Weiner

Ken hunched over his notebook, scribbling equations down as he leafed through his chemistry book. He flipped back through some of the pages and read for a minute. Finally, he began to write again. When he appeared to be finished writing, he tapped his neighbor on the back, pointed to something on his paper, and gave a "thumbs up" signal.

At this point, his English teacher asked him to share what he had written. Immediately Ken leaned forward, looked at the other students, and quickly began to share his writings. Ken wasn't in trouble for working on chemistry during the first ten minutes of English class. In fact, he was completing a writing assigned by his tenth grade English teacher.

Ken and his classmates in the Gifted and Talented section of sophomore English were working on a long-range assignment, writing in a learning log two days a week on the class that was giving them the most trouble. This presented problems for all the students. Only the nature of the problem differed from student to student. For most, the question was which of their classes was giving them the *most* trouble. They had to decide in which course they could benefit most from their writing. For a smaller group of gifted students, the problem was that no one course gave them all that much trouble.

Selecting a Subject

To help in choosing, students did a free-writing where they explored the possibilities of a learning log in their classes. After a 15-minute writing session, students shared their work. For many, it was risky to divulge that they were having difficulty in at least one of their classes. But to their surprise, several students wrote about courses they had in common, and it turned out that many students were suffering in silence about chemistry, biology, trigonometry, or French.

What followed was a half-class period devoted to students discussing individual problems—"I don't know why we have to memorize the equations" to "How do you know which function to perform first?"

Sometimes, a student attacked a teacher: "I can't ask questions in biology because Mrs. B berates me!" Even though several agreed, I cautioned students not to criticize or condemn individual teachers. The assignment wasn't intended to promote personal attacks but to focus on a way to learn subject matter.

How to Write a Learning Log

After the initial writing and the discussion of course problems, students were ready to turn to improving their knowledge in a particular course or at least to improving their grades. Their first learning log entry included responses to the following questions:

- What course will you focus on?
- What specifically is giving you trouble in the course right now?
- What content is the course presently focused on?
- What kinds of tests are given (objective, essay, etc.)?
- What grade did you get on the last test?
- What did you do right on the test?
- What gave you trouble?
- When is the next test and what will it cover?

This free writing helped students focus on their need to improve and on specific areas for improvement. It also tied the content problem areas to the specific kinds of testing and evaluation in a particular course. After this, students had a set of learning concerns and a deadline (the date of the next test or quiz) for improvement.

I then suggested that subsequent learning logs would follow these guidelines:

Audience: an imaginary friend that missed class that day

Content: major terms and concepts covered during class lecture, including specific examples and explanations, and a discussion of what you know and what you don't know (Paik, p. 108)

Format: freewriting with no particular concern for grammar, spelling and punctuation

Quantity: fill at least one whole notebook page twice a week

Time: twice a week during the first ten minutes of English class

Evaluating Learning Logs

During the eight weeks of learning logs, students shared writings with other students in English class. Students gave advice on course material and sometimes even made suggestions for students who shared the same problem course. This writing and sharing provided motivation for students to keep writing. After all, writing the entire 16 pages of a learning log the night before it was due was unproductive in using writing as a learning tool.

Finally, when students turned in their learning logs, I graded them primarily on quantity while skimming pages for spot-checking content. When the assignment was made, I spelled out the evaluation criteria. To get an *A*, students had to have 16 whole-page entries, and entries would have to deal with learning their subject matter by following the guidelines for content. For a *B*, students must have 15 pages; for a *C* 14 pages, and so on. So students knew exactly what was expected and what they needed to do to get their grade—a high priority with most high school students.

Since I'm an English teacher, I found it difficult not to mark grammatical and spelling errors, but I reminded myself that the assignment was to help students use writing as a tool for learning. The focus was on students making sense of what was happening in a course that was proving difficult. If students were to focus on learning and were not to be hampered by the packaging of their ideas, I had to agree not to penalize them for cosmetic errors. There were other opportunities to work with students on form and formality.

Student Reflections on the Use of Learning Logs

While student grades on learning logs proved to be high, this doesn't reveal how successful learning logs were for the students. In a final entry, students reflected on the merits (or lack thereof) of keeping learning logs.

As one sophomore noted:

> From this learning log I seemed to have gotten more out of each biology class because I had to write down what I learned. I better understood the material which helped me in this class overall. I view the class differently and I see how biology is important. . . . My grades are still the same but I better understand the material that is covered and am learning more from this class.

An algebra student made some discoveries about both her attitude and her study habits.

> I think that now I know that I should get help from the teacher when I need to, and ask questions more often. I was almost afraid to before. Also, I don't just think about math only in algebra anymore. I don't just turn my brain to the algebra mode for one hour while I'm in there, and turn it off for the rest of the day. Now, as I write the learning logs, I make myself think about what I am doing in class, so that I work out problems in my head, or try to solve something that I don't understand by myself.

A geometry student wholeheartedly endorsed the use of learning logs.

> My learning log has helped me in several different ways. I tend to learn better if I write the information over and over again. I wrote about problems I was having with different chapters which helped me when I reviewed for tests. I knew what sections to spend more time on from my logs.
>
> A couple times when I was doing my homework, I ran across problems that I could not answer. When I wrote them in my learning logs and solved them, I found out that I had figured out how to work out that particular problem.

Finally, the ultimate testimonial from another algebra student.

> My grades have gone up a noteable difference since the beginning of the quarter. I don't know whether or not it has been from doing these journals, but I must say that it hasn't hurt.

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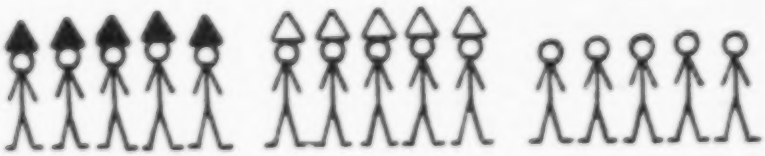
Half of Two Thirds is One Third

There are 15 boys.

$\frac{2}{3}$ of them are wearing hats.

$\frac{1}{2}$ of the hats are black.



$\frac{1}{2}$ of $\frac{2}{3}$ of the 15 boys are wearing black hats.

$\frac{1}{2}$ of $\frac{2}{3}$ of 15 is 5.

5 boys are wearing black hats. The 5 boys in black hats are $\frac{1}{3}$ of the whole group. $\frac{1}{3}$ of the boys are wearing black hats.

$\frac{1}{2}$ of $\frac{2}{3}$ is $\frac{1}{3}$.

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STUDENT LEARNING JOURNALS: CREATING THE HABIT OF REFLECTION

An excellent strategy for improving students' power in mathematics is to teach them to maintain a journal of mathematics learning. Teachers should help students form the habit of reflection by asking them to write every night about the meaning of what they are learning. It takes about two months to enable students to use this technique.

Rethinking and verbalizing new ideas clarifies and extends understanding. This active refection, several hours after class, helps students transfer new concepts into long-term memory, increases their responsibility for their own learning, and helps them identify and communicate their insights and confusions.

Help to create and emphasize success by regularly asking students to face the question "What have I learned today?". Each night's journal entry should include:

Today I learned about (or how to)_____.

The most important idea I learned today is:_____.

A question I have about what I leared today is: _____.

Something I learned today about thinking is: _____.

A way that I will use these ideas is: _____.

Processing new learning in this way will help students clarify their understanding of new concepts, and will help the teacher identify areas in special need of review and reinforcement. It will also help the students focus on the task rather than their own anxiety, and will help them appreciate their own progress. The teacher should provide positive feedback, stressing that each student has improved or learned something new.

Students may keep their journals in a separate book or notebook section, or write each night's entry on the same page as the rest of the homework assignment. Strategies for reviewing the assignment include collecting the journal on a weekly or daily basis, collecting the complete journal from specied students (on a rotating basis), assigning students to post journal entry selections on the chalkboard for class discussion, using class time to allow students to read and react to each other's entries in small groups. If small group sharing is used, the teacher may wish to ask each group to create a single entry, combining the ideas of all members, or may ask each group to respond to the question(s) posed by another. The teacher may involve the class in deciding the best way to structure this part of classroom life.

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HOW LEARNING JOURNALS HELP STUDENTS DEVELOP GREATER POWER TO SOLVE UNFAMILIAR PROBLEMS

First, try this problem yourself:

> In New York City, February was the shortest month in 1986. What was the second shortest month?

To clarify: "Second shortest" refers to the amount of time in the month. The characteristics of the names of the months and their alphabetical and chronological sequence have no bearing on the answer.

After this problem has been solved, student learning journal entries might include the following:

In solving this problem, I had difficulty because...
When I heard the word "short", I thought, "having fewer days". This is only one possible meaning of the word in the problem. When I thought about it longer, I realized that the word could also mean....

This is what the problem taught me about thinking skills:
I have to be careful not to take a narrow meaning of any word in a problem. When I do this, it blocks me from seeing the real solution. I should read the real problem which is given, and not change it into a different one by the way that I read it.

If I get stuck on a problem, it might help to go back to the beginning and review the way I thought about it, step by step. This way, I might find a good point to go off in a different direction.

By reflecting on the experience of solving this problem, the student may have an insight about a common stumbling block to success, and a way to avoid it.

By verbalizing and recording this insight, he or she becomes more likely to retain it.

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THE LEARNING LOG

Research has shown that one of the reasons youngsters experience long range failure in mathematics is their inability to internalize concepts. This manual is designed to illustrate how to develop concepts on a daily basis, but its effectiveness will be short range unless it is supplemented by pedagogical practices keyed to transfer learning from short-term to long-term memory.

One effective tool for assisting youngsters to internalize new ideas is THE LEARNING LOG. Youngsters are initially taught how to complete each of the four entries in the log, one at a time, in class; the effect is felt when they are able to complete the log not in class. but at home.

Phase I: Youngsters are asked to identify the most important idea explored that day in class (or during a specific lesson in class), They must be taught

- NOT to copy the aim of the lesson;
- NOT to copy statements from a textbook;
- NOT to copy statements from their notes;
- NOT to identify a fact or rote process as an idea;
- to think about the lesson for five minutes before writing;
- to use their own words and modes of expression.

Classroom practice and discussion on at least three occasions in small groups with student interchange and intergroup sharing is essential before phase I is assigned for homework.

Phase II: Youngsters are taught to improve their own questioning skills so that they can begin to become self-directed learners. Model questions are provided to youngsters as samples.* Good student questions are written on the blackboard and annotated (preferably in color). Good teacher questions are written on the blackboard and similarly annotated with explanations of why they are good (they expanded comprehension, they explored analogies, they developed an application, they explored how an analysis should be made, they explored criteria for evaluation of solutions to problems, etc.). Not every question is written on the blackboard, but a sufficient number are written so that students learn to take notes which reflect question and answers and not answers only.

After preliminary instruction on improving student questioning and after students have been doing phase I at home for at least two weeks, they begin to do phase II of their learning log at home. A question I would like to ask about what I learned today is:_____.

Note: Sharing of phase I ideas and phase II questions in class the following day is an important part of the transfer of concepts to long-term memory. Students should be asked to evaluate ideas by identifying the best phase I responses and telling why they are best and by identifying the most helpful phase II questions and telling why they were helpful.

Phase III: After phase I and phase II have been firmly established homework routines, students should be helped to think about ways in which their most important idea can be used. Why have they selected the idea as most important? They must identify some way in which they can use the idea other than a way directly illustrated in class.

Phase IV: The final phase of the LEARNING LOG requires students to think about thinking; i.e., they become metacognitively aware. Have they used analogy to solve problems that day? Was comparison and contrast employed to clarify thinking? How was pattern perception used? Once the children have names for problem exploration strategies, phase IV helps them to see the relevance of learned strategies to actual situations in or out of school.

Note: When you note that a few children are responding well to the phase III and phase IV entries at home (after brief classroom introduction), they should be encouraged to share and explain their applications and insights.

The entire successful introduction of the LEARNING LOG as a nightly component of homework may take two months. It will be time well spent for you and your class. What students think about at home (after a time lag) and write about they will be able to recall far more efficiently and they may be able to apply more effectively in new contexts.

Teachers should try learning logs for themselves in conjunction with courses being taken and complex books they are reading. If they work for you, shouldn't you share the success with the children you teach?

"WHAT IF...?"

Similar Problems to Explore:

1.

2.

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MATHEMATICS JOURNAL

Student: _____

Date: _____ Place: _____

Problem or Subject: _____

Specifics: _____

The most important idea I learned today is:

. A question I have about what I learned is:

3. Something I learned tpday about thinking is:

4. A way that I will use these ideas is:

LESSON PLANNING

LESSON PLANNING IN MATHEMATICS

1. The math teachers should prepare a detailed lesson plan for each lesson.

2. The parts of a lesson plan are: (a) Aim (b) Performance Objectives) (c) Do Now (d) Motivation (e) Development (f) Applications (g) Summary

3. The aim should be stated clearly and simply and written in question form. It should be elicited from the pupils. It need not, however, be elicited at the very beginning of the period.

4. The teacher must recognize what previous learning pupils will need as a foundation on which to base the principle to be taught in the new lesson. The do-now questions should include pre-requisite skills, defintions or procedures needed for the new lesson.

5. The motivation should be brief and to the point. It should lead to a statement of the aim of the lesson and a desire to achieve it. The motivating problem should be simple enough to be solved by the class after the development of the lesson is completed.

6. During the development of the lesson, the teacher should:
   a) maintain control of its pace and direction
   b) avoid dominating it, but promote widespread pupil participation and inter-pupil relations, as much as possible eliciting rather than telling correct answers.
   c) make use of proper questioning techniques, asking questions which are clear and unambiguous, including a number which will stimulate thought and evoke sustained responses by pupils, as well as others which will help pupils recognize shortcomings and errors in some of their answers.
   d) think through the statements that will be made, the questions to be asked and the answers expected from the pupils as the lesson develops.
   e) try to anticipate incorrect answers or unexpected questions to see where they may lead, so that the lesson may not be misdirected. An unexpected wrong answer or a surprising question can easily upset a lesson for the entire class.
   f) include the wording of statements and of questions in the lesson plan.
   g) elicit from pupils complete and accurate statements, suitably and clearly expressed, employing technical vocabulary correctly and intelligently.
   h) check that pupils understand the principle used to obtain correct answers submitted using the lesson.
   i) see that all relevant material is not neglected, and that irrelevant matters do not intrude.
   j) present a sufficient number and variety of situations before calling on pupils to formulate a generalization.

7. The teacher should prepare in advance a statement of the generalization to be reached by pupils in the lesson and of the question he/she will ask in order to elicit this generalization from the class. The statement should be in language that is clear and easily understood by pupils, and mathematically correct. Any technical vocabulary employed should be adequately reviewed. By skillful questioning, the students themselves should be helped by the teacher to formulate the generalization.

8. One of the first applications of the principle learned during the lesson should be to solve the motivating problem previously mentioned. Other applications of this principle should be presented to the class.
   When pupils submit correct solutions, the teacher should see that in each the

principle used to obtain them is emphasized. The teacher should prepare in advance a large number--more than he thinks he may need--of carefully graded examples which require the use of this new principle.

9. A summary of what was accomplished during the period should be elicited from the students. The summary is a check upon whether the aim of the lesson has been achieved.

10. The presentation of the new lesson should not be at the expense of reviewing the homework. The teacher must see that there is time for both.

11. It is also preferable that a teacher plan to cover more than can be covered in a period rather than find a void in the lesson for even a few minutes.

SIGNIFICANT ASPECTS OF A MATHEMATICS LESSON

1. "Do Now" (Warm-Up) or (Seatwork)
   a. Should be related to the new lesson, not just an arbitrary exercise i.e. should include pre-requisite skills, definitions, or procedures needed for the new lesson of the day.
   b. Should be written on the board at the beginning of the lesson.
   c. Should serve as a transition between previous learnings and new.
   d. Should take no longer than five minutes to review.
   e. Should be short and attainable by students within the first few minutes of the period.
   f. Can be used to provide writing experience where appropriate by asking students to write definitions, procedures, explanations, etc.
   g. Does not have to be used in a math lesson where the lesson begins immediately.

2. Motivation
   a. Should create a need to know ("in order to solve this problem we must learn this new technique...") or a desire to know ("Why does this card trick work?")
   b. May lead from a problem or apparent paradox
   c. Is short
   d. Yields the aim
   e. Is referred to in the summary

3. Aim
   a. Should be in the form of a question.
   b. Can be easily measured for achievability.
   c. Should be written on the board and left for the entire period.

4. Homework
   a. Should be cyclical (i.e., examples from a month ago, week ago, day ago)
   b. Should be reasonable in length
   c. Should take at most 10-12 minutes to review, preferably less.
      1. Student should "carry" review.
      2. Solutions and questions should be written out on students' work.
      3. Board responsibilities should be assigned in advance.
      4. Can employ visual devices (overhead, colored chalk, etc.) to save time and promote clarity.
   d. Should be numbered.
   e. Should be assigned at the start of the period and written on the same area of the board or be included in a homework assignment sheet.
   f. Should include RCT math or regents review questions.
   g. Should be solved by teachers in their lesson plans.
   h. Should be checked or collected.
   i Can be referred to appropriately in the new lesson.
   j. Provides pre-requisite skills for the new lesson.
   k. May be reviewed at the beginning or the end of the period.
   l. Should include a reading assignment.

5. Development
   a. Concepts should be taught along with techniques i.e., the "why" along with the "how"
   b. One new thought should be developed at a time in the sequence of logical analysis.
   c. The problems used as models should be well chosen and graded in diffculty. i.e., they yield the desired concepts and skills to be studied.
   d. The joy and beauty of the mathematics should be revealed.
   e. Concrete models, diagrams and other visual devices should be employed.

f. The board work should be detailed, neat and written from left to right on the board.
g. Students should share a mutual discovery with the teacher.

6. Questioning
   a. Skillful questioning serves to direct activity:
      - to provoke discussion which will extend knowledge and awareness
      - to stimulate thinking
      - to measure accomplishment by testing information and understanding
      - to seek evidence of reflective thinking and judgments based on proof.
   b. Skillful questioning should keep the student more alert, more critical, more active. Therefore, frame questions of graded difficulty in order to maintain interest, and appeal to the varying abilities of students. Distribute questions, calling on non-volunteers as well as volunteers. Handle the erroneous answers of the weaker students without belittling their attempts. Encourage topical, sustained answers.
   c. To stimulate critical thinking, questions should evoke sustained answers. Therefore, it is poor practice to ask too many questions or to use only factual questions.
   d. After asking a difficult question, it is wise to pause before calling on a student. Such a pause will provide time for thought and discourage the quick, glib answer. The pause also provides time for students who think more slowly to react.
   e. Once a student begins a developed answer, permit him to finish without interrupting him.
   f. Since at all times responses should be directed to the class rather than to the teacher, other class members are free to challenge the accuracy of these responses. (Such automatic procedures result, of course, from constant, patient training in the early weeks of the term and from reminders given constantly throughout the term.) Where an individual--teacher--individual recitation is used, general student participation is impossible, and lack of respect for others as well as inattention (day-dreaming, talking, etc.) will result. The teacher has to keep the group in the picture even when dealing with an individual problem.

   g. Criteria for judging questions asked in the classroom

      1. They should be simple and direct in language.
      2. They should be specific in directing attention to the particular problem under consideration.
      3. They should follow a planned order which develops the train of thought in a logical sequence.
      4. They should place the burden of thinking upon the student (and, therefore, should be relatively few in number).
      5. They should be so framed that the answer may not be a simple 'yes' or 'no'.
      6. There should be included in the lesson questions which require students to draw conclusions.
      7. Questions should be adapted to the varying levels of ability in the class.
      8. They should be sufficiently difficult to arouse effort, yet not so difficult as to preclude a response.
      9. There should be a proper balance between fact and thought questions.
      10. Questions should not be repeated. (If you must have one repeated, have a student repeat it).

7. Applications and Drill Problems
   a. Should be graded from simple problems to complex
   b. Should be reviewed on the board by students in the class
   c. Should provide enrichment for talented students
   d. Should be pre-planned (and solved completely in teacher's lesson plan).

8. Summary
   a. Should be elicited from students
   b. Often yields a "recipe" type of problem-solving procedure. Should include key concepts as well.
   c. Can be generated from a "tour" of board work from left to right.
   d. Should refer to the aim.
   e. A medial summary, elicited from the class, should be provided prior to application and/or drill.
   f. A final summary elicited from the class, should be provided after the application problems.

9. Miscellaneous
   a. Teacher should circulate around the room monitoring notebooks and assisting students with individual problems.
   b. Teachers' board work should serve as a model. They should use a ruler, be neat and write out full sentences. Diagrams should be easy to read.
   c. Teachers should have worked out solutions to all homework problems and problems used during the period.
   d. All problems used should not be test oriented.
   e. Timing of a typical lesson:
      5 minutes--"Do Now"
      5-10 minutes--Homework review
      12-20 minutes--Development
      7 minutes--Drill and/or applications.
      2 minutes--Summary

THE USE OF THE "DO NOW" IN A MATHEMATICS LESSON

I. Why do we give "do now" problems?

A. To review pre-requisite skills, concepts, vocabulary or procedures that will be needed for the new lesson.
B. To provide opportunities for integrating writing in the mathematics class.
C. To set a routine of students beginning their work as soon as they enter the room. This also sets a tone that the entire period is spent on instruction and not irrelevant activities.
D. To provide the teacher with an opportunity to handle clerical matters and yet not interfere with student learning. This is only one of several opportunities that the teacher has to handle such things as taking attendance without having students unoccupied in a learning activity.

II. What are some of the characteristics of a well chosen "do now?"

A. It reviews pre-requisites skills, concepts, vocabulary or procedures needed in the new lesson.
B. It is within the ability of the class to answer.
C. It can be copied and completed by the class in no more than the first five minutes of the period.
D. It can be used as an opportunity to provide mathematics students with writing experiences.

III. What should we avoid when we use a "do now?"

A. Don't allow it to take too much time.
B. Don't allow it to become a review lesson. Review lessons, like all lessons, must be motivated and developed. The "do now" should NOT become the lesson.
C. Don't include the new topic in the "do now" assuming that the students will know how to do it. If the students can answer questions on the new topic, perhaps, you do not need to teach that lesson.
D. The "do now" is not the motivation, although it may contain part of your motivation.
E. The "do now" is not independent from the rest of the lesson. There should be a transition from the "do now" to what follows. It should be referred to at some point in the lesson. If it is not referred to, perhaps it was not needed.
F. Don't use a "do now" if the lesson does not call for it. A lesson can start immediately without a "do now".
G. Don't be redundant. If your homework assignment has reviewed your pre-requisite skills, you need not do it again in a "do now".

FROM LESSON PLANNING TO PERFORMANCE

A. A lesson may not be successful despite the fact that the plan included a worthwhile aim, a challenging motivation and thought-provoking questions. The following are suggestions to help you insure success in the execution of your daily plans:

1. You must be prepared to answer questions that may be raised by the pupils. The answering of such questions may require a knowledge of your subject that goes beyond the content of the textbook. A potent motivation for learning on the part of the pupils can be the scholarship of the teacher.

2. You must be prepared to develop the lesson along a path different from the one you set down in your plan. This way come about because of the answers that are presented in response to your motivation. The class may suggest a better aim than the one you have prepared.

3. You must make provision for the active participation of the greatest number of pupils in each lesson. Do not permit a few students to dominate or by watching demonstrations. To insure the greatest amount of participation it is suggested that:

   a. You establish the rule that hands be raised only when you request volunteered answers or when a pupil wishes to ask a question.

   b. You call on the slow, passive and shy pupils to repeat answers, to answer simple questions, and to evaluate answers.

4. You should make the entire period, not only the motivation time, a stimulating and challenging experience for the class. Suggested procedures to realize this goal are:

   a. Sustain the motivation throughout the period.

   b. Employ questions that stimulate thinking and that produce interaction among the members of the class.

   c. Use demonstrations and experiments that excite the interest of the pupils.

5. Make certain that, throughout the period, understanding and retention are achieved on the part of the pupils. You should recognize that answers may be inaudible, fragmentary, or poorly-phrased; that members of the class may be inattentive at times; and that all students may not comprehend answers or board demonstrations. The following suggestions are given to insure understanding and retention:

   a. Insist that pupils speak in an audible tone to the majority of the class. If you feel that an answer has been inaudible to some members of the class, have it repeated by a student.

   b. If an answer lacks details, if it does not cover all facets of the problems, or if it is totally or partially incorrect: call on other pupils to supplement or to evaluate the answer. If necessary, pose another question that aims at soliciting additional information.

   c. If, after calling on a few pupils, you are unable to elicit the desired response--rephrase the question. If the response is still not forth-

coming, present the answer yourself. Do not waste valuable teaching time in an attempt to secure an answer to any one question.

d. Prepare pupils for the observing of experiments and of board demonstrations. During the execution of experiments and demonstrations ask questions and solicit questions.

e. Call on a student to provide a summary after a number of answers have been presented in response to a particular question.

f. Include drills, medial summaries and final summaries and applications in every lesson.

PRACTICE MAKES PERFECT

A. Many roads lead to Rome. There is no one correct method to teach a particular topic or to achieve a particular aim. The plan used successfully by one teacher with a particular class may not achieve the same results when used by another teacher. You will, as a result of your own planning, discover the most effective methods to be employed in teaching each topic with different classes.

B. More effective teaching on your part will result in better pupil achievement, better class discipline and an easier and a more satisfying teaching day for you. To improve on your lesson plan it is suggested that:

1. You critically evaluate each lesson in terms of the interest of the students and of the learnings that took place. If your plan has not proved effective during the period that you taught it, try to revise it before you teach the same lesson to another class.

2. Secure suggestions from veteran teachers in your department for the teaching of particular lessons.

3. You pre-plan lessons with your supervisor.

4. You observe more experienced teachers.

5. You re-write your lesson plans each year. Classes change; the world changes. An aim or a motivation that is timely in one year may have no appeal a different year.

6. You consult frequently with your subject supervisor.

Practice will not make you a perfect planner or a perfect teacher. There is no such person. What it will do is make you an interested teacher, an inquiring teacher and therefore a better teacher. The secret of your success will lie in your own efforts toward self-improvement.

Grouping for Instruction

The mathematics teacher of a Grade 4 or 5 class has a variety of grouping strategies available as aids to instruction. When judiciously used, these different kinds of instructional groupings sharply focus the instruction, providing pupils with precisely the instruction that they individually need at a particular time Effective grouping increases the level of interest and involvement for pupils in productive pupil-to-pupil interactions, and assists the teacher in utilizing class time as efficiently as possible. There are three basic kinds of grouping.

Full-group instruction:

This is most effective when introducing a topic to the class for the first time. It involves class discussion, question-and-answer exchanges between pupils and teacher and contributions from all class members. These contributions may be in the form of oral answers to questions, work at the chalkboard, the formation of hypotheses, the creation of student-designed problems, student-to-student communications regarding previous student contributions, and written work at students' desks. Full group instruction may be devoted to the development of new concepts, the practice of recently learned concepts, the review of previously learned concepts, or a combination of all of these. Full-group instruction is used when the teacher has determined that the needs of all the members of a class are the same at a particular time.

Small group instruction:

Soon after the introduction of any new topic -- indeed, soon after the initial assessment of pupils' skills at the beginning of this program -- differences in pupils' needs will become apparent. Not all of the pupils in a class will need to continue working on a topic for the same amount of time. There will be groups of pupils whose skills with a particular topic are approximately equal, and these small groups are directed by the teacher to work separately on tasks appropriate for each group. A group may consist of two pupils or more. The creation of these groups and the process of setting them up, each with its own tasks and instructions, will involve the physical movement of pupils in the room. A flexible atmosphere should be established in which such movement is not only tolerated but is seen as an important part of the learning environment. One small group may be working on an enrichment activity consisting of a new board game, calculator activity or research project. One group may be a pair of students: one who needs help with a particular topic and another who has mastered it and is serving as a peer-tutor. Small-group instruction is an extremely flexible tool that can fully exercise a teacher's creativity. It is a dynamic process requiring constant evaluation and reorganization of the members of the class.

Individual instruction:

There will be times when an individual student's needs can not effectively be met by large-group or even small-group instruction. Perhaps

the student has been absent for a particularly crucial series of lessons, or is a new arrival to the class, or is simply the only one in the class who, for one reason or another, has not caught on to a particular concept. At this point, while the rest of the class is engaged in working on a class-wide activity or involved in their small-group activities, the teacher intervenes on a one-to-one basis to provide this pupil with the personal instruction he needs. Once the pupil's skill is brought up to a level with other pupils in the class, he may be made part of a small group. Peer-tutors, students with strength in a particular skill, may also be used to provide individualized instruction.

Cooperative Small Groups: A Method for Teaching Problem Solving

By **Kathleen Gilbert-Macmillan** *and* **Steven J. Leitz**

If the objective is the development of abilities and skills which are problem-solving in nature, the least efficient discussion is superior to most of the lectures.

Benjamin Bloom (1953)

Learning how to solve problems is fundamentally different from learning basic computation. Yet we often try to teach problem solving with the same methods we use to teach addition and subtraction—short lectures with perhaps a demonstration and lots of individual seat work on practice problems. Children's thinking may be limited by these methods when the problems require higher-order thinking. In our work with upper elementary aged children, we have found that organizing the class into three- and four-member groups that are trained to work cooperatively can provide an instructional setting effective for improving students' problem-solving skills.

Word problems generally require more complex thinking processes than those required for routine exercises. To complete an exercise, a student applies an algorithm that leads to the solution. The approach to solving a word problem is less direct. Several steps seem necessary. The student needs to discover what information is relevant, plan a way to manipulate the appropriate information, carry out these manipulations, and evaluate the solution by looking back to the statement of the original problem.

The development of such skills is particularly well suited to the cooperative small-group environment. Interaction among children is one of the chief motivators for the learning process. Children's thinking and insight are stimulated by explaining their solution processes to others, as well as by the ideas and questions offered within the group. They come to examine a problem more objectively and begin to use new perspectives when they are exposed to different points of view.

When good problem solvers try to understand a problem, they do not immediately translate the words into calculations to be performed. Preceding that is a period of reflection when they try to understand the problem. Then they proceed through a process to arrive at an answer.

Students can refine and practice their mathematical language.

Small groups of children working together have many opportunities to express themselves and to gain experience in using mathematical language correctly. When a group of four students in our fifth-grade classroom were struggling with a problem asking for the distance around a rectangular garden, the following dialogue took place:

John: What's a rectangle?
Dave: Like this—three sides [draws a triangle].
Anita: No. *Four* sides. Like this [draws a rectangle]. A rectangle has four sides. You made a triangle.
Dave: Oh, yeah. A rectangle. . . .
John: Four sides.

When students refine and practice their mathematical language, they are acquiring active skills that are necessary for problem solving.

Besides developing their use of mathematical language, children in small groups working on mathematics word problems also improve their overall oral communication. Consider the following dialogue we tape-recorded during a problem-solving session when the problem was this:

> Juan ate 1/3 of a papaya and his friend also ate 1/3 of the papaya. How much of the papaya was left?

Alfred: What's a papaya?
Raymond: It's a big thing like that.
Lisa: Say it's a pie.
Raymond: Some kind of veggie.
Elise: Yeah, let's make it a pie.
Alfred: No, it's a fruit. I know it's like this.
Raymond: It's like an oval.
Alfred: Yeah. Like a watermelon, kind of.
Raymond: And it has little black things inside. Like seeds. And it's orange.
Elise: Ohhhhh. . .that!
Alfred: Papaya.
Raymond: Yeah.
Alfred: They taste like, ummm.
Elise: Melons.
Alfred: Yeah.
Raymond: Overgrown melons.

As soon as the notion of a papaya was

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established, the group proceeded to solve the problem.

In small groups children can be dissuaded from inefficient or inappropriate problem-solving strategies. When Lora was given the problem

> I am thinking of a number. My number is eleven less than eighty-three. What is my number?

her peers gently rebuked her random "add, subtract, or multiply" approach:

Lora: First you *times* it, ya *plus* it, then ya . . . *take away* it and then you . . . see which one you want. [Pause.] So it's either 72 or . . . 94 or. . . .

Darby: Lora!

Several studies have found that children spend more time on task when they work in small groups than they do when they work individually in large groups. Slavin (1978) has also found that students usually become more involved and prefer instructional activities with small groups. However, these results are not *automatic*. Merely placing children in groups of three or four does not, in itself, ensure that they will become better problem solvers. Without carefully structured tasks and group training, interpersonal relations can deteriorate. Rates of individual participation can be highly unbalanced, and the group may be dominated by a single student while the others are nearly silent.

A number of factors affect how well small groups function. The composition of a group, for example, tends to have some influence on its processes. The number of students in the group and their perceived status, both social and academic, have been found to make the most difference. How the group members are chosen—whether by the teacher or by one another—as well as their ability, gender, and ethnicity seem to have less effect. We have found that three to five students in each group is necessary to achieve maximum participation; our most successful groups usually have four members.

How other class members perceive the student's ability and "social standing," that is, the status of each group member, plays a large role in the functioning of small groups. Although status differences cannot be totally erased in small groups, they can be minimized through training. See table 1.

Table 1
Key Elements in Training Small Groups to Work Effectively

- Clearly specify the group tasks.
- Point out that each member of the group has unique abilities. Some are better at reading, some at computation, others at drawing diagrams, and others at managing the group's procedural activities (including monitoring the taking of turns and recording answers).
- Emphasize that the students can learn from each other and should share responsibility for each one learning how to solve each problem.
- Tell the students specifically what kinds of behavior are important in their small groups—for example, listening carefully when other group members are speaking and making sure each member participates frequently. The groups will operate more effectively if they are specifically encouraged to ask one another questions, explain their ideas in detail, draw pictures, and debate their answers.
- Although training is essential for the students to work cooperatively in small groups, it is best administered in brief doses with reminders between the problem-solving sessions.

The structure of the cooperative task must be clear and specific. The task design described next is an example:

1. Each student has a copy of the word problems, a pencil, and scratch paper.

2. A group answer sheet is provided. The group will give the teacher this paper at the end of the class period.

3. Students take turns reading the problems and writing down the group's answer. All students, however, are part of the solution process.

4. The group tries to agree unanimously on the answer. When two or more different answers are suggested, it is the group's responsibility to debate them, explain to one another, and try to reach a consensus. If the group cannot agree on a single answer, then the tentative answers should be recorded. Students are encouraged to struggle when an answer eludes them.

5. When the group has finished its problems, the teacher gives them a solution sheet with only the correct answers. The group checks its answer sheet for errors and tries to determine how to reach the correct solutions for the problems that were missed.

6. Some class discussion might follow the small-group work. The discussion can be stimulated by presentations of solutions. Individual students or a group can be called upon to make a presentation about a particular problem. Students from different groups might be asked to sketch diagrams or write down number sentences on the chalkboard that were useful in solving a problem. Good problem solvers recognize that multiple paths lead to correct answers.

The complexity of the word problems will depend on the background of the students, but generally the groups can focus better on problem solving when the computational requirements are less distracting. Typical textbook story problems are appropriate, although more intricate and realistic problems lead to valuable discussions and the development of practical skills.

Children need to discuss information that may seem peripheral to the problem itself.

Two final points need to be made about cooperative small groups as a method for teaching problem solving. First, children need to discuss information that may seem peripheral to the problem itself. Although we are often tempted to interrupt them and redirect their attention to the core of the problem, the wandering paths, blind alleys, and odd turns they choose to take are part of the experience of solving problems. Second,

managing a classroom of small groups of upper elementary schoolchildren working on word problems is not easy. The rewards cannot be reaped within a day or a few weeks. But given training, practice in cooperative small groups, and the time to question one another, pursue their own thoughts aloud, and struggle to agree on solutions to mathematical word problems, children will become better problem solvers.

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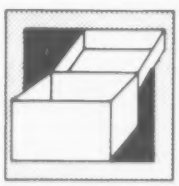

From the File

Computation

MULT-I-ADD BINGO

Objective: To strengthen addition, multiplication, and estimation skills; to strengthen visual perception in children who are perceptually impaired

Materials: Old bingo cards

Directions: Students are given the cards with some pattern of numbers circled on them. They estimate what the sum (or product) will be, compute the result, and compare it with their estimate. The patterns on the cards can be lines of numbers that form bingo, or they can be other patterns.

14	19	32	46	66
3	25	41	58	75
12	26	FREE 35 SPACE	55	74
7	17	45	47	67
13	29	31	49	70

From the file of Marilyn L. Marshall, Thomas O. Hopkins Middle School, Burlington Township, NJ 08016.

—Readers are encouraged to send in two copies of their classroom-tested ideas for "From the File" to the Arithmetic Teacher for review—

"COOPERATIVE LEARNING GROUPS"

Prepared by Members of the Mathematical Editorial Staff
of Merrill Publishing Co.

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INTRODUCTION

Teaching lessons can be structured three different ways: individually, competitively, and cooperatively. All three of these are important depending on the objective of the lesson. The teacher needs to decide when each method should be used.

In individual learning, student success or failure is unrelated to that of their peers. The student is concerned only with his or her own learning and is unconcerned with the learning of others. In competitive learning, students are matched against one another to see who can be the most successful. In cooperative learning, students work in small, structured groups to achieve a common goal. The success of the cooperative group depends on the positive interaction of the members.

While each of these strategies are important and can be used constructively, research has shown that by using cooperative groups, students not only learn subject matter more effectively but also develop valuable interpersonal skills.

WHAT IS COOPERATIVE LEARNING?

Ideally, in any situation, the teacher must have well-defined objectives of student learning and a clearly stated evaluation method. This information should be presented before the lesson begins so students know what is expected. While this is also true of cooperative learning, the main emphasis is that students within a group help one another achieve a common goal -- learning the material.

Cooperative learning is more than just putting students near each other or sharing materials. The teacher divides the class into heterogeneous groups of two to six students. Once the task has been defined, the method of evaluation is explained. A variety of evaluation methods should be used, such as selecting one paper from the group or averaging the grades of each student for a group score. Evaluation of this kind reinforces the idea that each member is responsible for the learning of all members of the group. As a way of providing closure for each activity, students discuss what they have learned and how they have worked together.

Four essential elements must be included in cooperative learning: positive interdependence, face-to-face interaction, individual accountability, and interpersonal and small group skill.

Positive interdependence is developed by carefully structuring learning goals. The result is students becoming concerned with the performance of all group members. The optimum level of interdependence is an attitude of "We're all in this together".

Face-to-face interaction is established by such simple things as sitting on the same level so that eye contact, listening, and discussion can easily take place.

The purpose of any teaching strategy is to learn the material. In cooperative learning situation, individual accountability is present when each student's progress is assessed so they know their own strengths (how they can help others) and weaknesses (how others can help them). By highlighting strengths, students

also gain self-esteem which enables them to interact with others instead of riding on another's coattails.

Too often, when we put students into traditional groups, we assume social skills are already present. A key to effective cooperative learning is that small group skills are actually taught to the students. One important aspect, which is often overlooked, is processing. This teaches students to analyze how well the group is working together and to make changes that are necessary for improvement.

BENEFITS

Studies have proven that the use of cooperative learning groups postitively effects students, both academically and socially. Academically, higher order thinking skills are developed through the use of problem-solving strategies. Students spend more time on task and there is a greater retention of knowledge. The end result is that students are more motivated and enjoy learning.

Socially, barriers are broken down, allowing students to accept others regardless of ethnic background, differences in physical or mental abilities, or social status.

Students also learn a great deal about their individuality and self-worth. As students interact, their confidence and self-esteem are enhanced, allowing them to voice their opinions freely without fear of negative peer pressure.

Improved social skill are evident outside of the group -- on the playground, in the lunchroom, at home -- with peers and adults. Each group begins to model a society where opinions and choices affect not only the individual but the group as a whole.

There are also benefits for the teacher. Less time is needed for classroom management because students become less teacher dependent as confidence grows in their ability to solve problems. The teacher's role becomes one of a facilitator of learning. The curriculum can be covered more thoroughly as students learn to discuss and analyze processes rather than focusing only on getting the correct answer.

TRAINING

A variety of materials is available to help teachers learn about cooperative group strategies. Attending courses or workshops is extremely beneficial and can aid in implementation. It is suggested that teachers interested in using this teaching model consider working with other educators in their building. This provides ongoing support, reinforcement, and feedback.

GETTING STARTED

As with any new experience, groundwork needs to be laid. Skills need to be taught and built upon. In any cooperative activity, three basic stages need to be followed:

. setting the standards
. performing the activity
. processing and debriefing

Before any activity begins, explain the task, means of evaluation, and behavior expectations. Post the rules and explain grouping arrangements. After the activity has been completed, allow students to express their reactions. While the time spent on each stage will vary, all are equally important and contribute to the success of cooperative learning.

Various techniques may be employed to ease both the teacher and student into a new method of learning. One method would be to use activities that draw from students' personal experiences. For example, a question of the day can bring out individual feelings during a classroom discussion (I feel angry when..., How would I change Santa's job? etc.).

Smaller groups could decide what ten things they would need to take to the moon for survival. Discussing topics that involve compromise brings out values clarification and prepares students to work more productively in a group situation.

While these activities are non-academic, they allow students to verbalize thoughts in a non-threatening way. They also provide an opportunity for self-awareness.

Another technique is to begin cooperative groups using an academic subject. Experts suggest that teachers choose a subject area with which they are comfortable. A review or maintenance would be a good starting point. Because students feel comfortable with the subject matter they are more likely to take an active part in the discussion. Using this method, one social skill at a time can be introduced and emphasized.

Whichever method is chosen, teachers should allow ample time for these inclusion activities. When it is evident that students are interacting, encouraging each other, and becoming interdependent, the teacher can expand the use of cooperative strategies to teach new academic skills.

IMPLEMENTATION

There are many components to consider when planning to implement cooperative groups.

Choose the lesson

Decide the outcome (general goal) -- to memorize, practice, discuss, investigate, or review. State the specific objective(s) students will accomplish.

Determine how introductory material will be presented -- lecture, homework, movie, etc.

Decide the time needed to complete the group activity. Allow enough time for set-up, work time, and processing. Gauge time according to the

attention span of the class.

Select the method of interaction

Teachers need to give students a basis for interaction in a variety of ways such as sharing limited materials or resources, or turning in one finished product. The use of different cooperative learning methods is another way to create this interdependence. The most widely used methods include: Student Teams-Achievement Divisions (STAD), Teams-Games-Tournament (TGT), Jigsaw, Learning Together, and Group-Investigation. The following chart highlights similarities and differences of each method.

	Student Teams Achievement Division (STAD)	Teams-Games-Tournaments (TGT)	Jigsaw
Suggested Subject Area	any subject best with set answers	any subject best with set answers	best with narrative material, especially social studies
Group size	4-5	4-5	5-6
Structure	teacher presents lesson students use worksheets to master material	teacher presents lesson students use worksheets to master material more competitive	each student important information meet in "expert" groups return to teach groups
Evaluation	individual quizzes group score based on individual improvement	students represent teams in academic games students compete with others of similar ability team score based on individual improvement	individually tested
Developers	D.L. DeVries, K.J. Edwards, R.E. Slavin	D.L. DeVries K.J. Edwards, R.E. Slavin	E. Aronson

	Jigsaw II	Learning Together	Group Investigation
Suggested Subject Area	Best with narrative material, especially social studies	any subject	best with science, social studies, or literature
Group size	4-5	2-6	2-6
Structure	students read all material (entire story, chapter, etc.) students assigned special topic meet in "expert" groups return to teach group	teacher presents lesson students work to complete task closest to pure cooperation	students responsible for deciding and organizing what and how they, learn students share knowledge with class most complex
Evaluation	individual quizzes add to team score	varies according to task grades often based on group performance	varies according to task
Developers	D.L. DeVries K.J. Edwards R.E. Slavin	D.W. Johnson R.T. Johnson	S. Sharan T. Sharan

Select methods for evaluation and reward

Use a variety of evaluation techniques to ensure individual accountability. Almost any method of evaluation is acceptable as long as criteria is established and students understand how they will be evaluated. Criteria can be adapted within a group to allow for individual differences.

Determine if or when rewards seem appropriate or necessary. When offered, rewards should be given to the group as a whole. Everyone earns the reward or no one does. Rewards can be verbal recognition, special privileges, or tangible items. Eventually, the learning should be self-rewarding.

Decide on group composition

Size of the group should be 2 to 6 students. Beginning with small groups (about 3) requires less organizational time, faster operation, and allows more "air time".

Selection of the group should be heterogeneous. Include students of all ability levels. Be sure each group has combinations of handicapped/non-handicapped, male/female, outgoing/shy, and various races. In the beginning stages, groups should be teacher-selected and not contain extreme differences in academic levels. As skills develop, vary the method of group selection -- choose different attributes, select randomly, invite student input, etc.

Length of time the group stays together varies from task to task. Let groups stay together long enough to establish unity and to be successful. Hopefully, a feeling of "team spirit" will develop. If this "team spirit" turns into a competition between groups, it may be a sign that it is time to change groups. If a group is having difficulty working together, it is important not to intervene or break it up. Let them continue to work as a group to solve their differences.

Arrange the room

Plan areas where groups will meet to the arrangement requires the least amount of furniture movement. Four or five students can easily cluster around one or two desks, or sit in a circle on the floor.

Group members should be close enough to be able to hear, see, and face each other. Of utmost importance, students within a group must be on the same eye level. This emphasizes that everyone is an equal and encourages each to participate.

Allow enough room between groups for easy movement. Space also serves as a noise buffer.

Set class rules and standards

Decide on rules for group work. Students may participate in determining rules. Examples might be:

Listen to others

Encourage others (No put-downs)

Use quiet voices

Stay with your group

Take turns

Help others

Correct until all agree

Keep the list simple and brief. More can be added if needed.

Explain the task and method of evaluation

State the objectives in terms students will understand.

Explain procedures to be followed and desired outcome(s). Showing examples may help to clarify the assignment.

Explain the evaluation method that will be used for the task.

Answer any questions students might ask. Ask students specific questions to ensure that all understand the assignment.

Student roles

Some groups may function better if roles are established within the group. Roles serve a dual purpose: to reinforce that everyone is an important part of the group, and as a means to keep the momentum flowing.

Possible roles:

Leader/Coordinator - keeps group on task

Reporter - summarizes group's work to class or teacher

Recorder/Secretary - writes group's answers

Reader - reads directions

Observer/Evaluator - keeps track of individual participation

Encourager - uses positives reinforcement

Whether roles are specified or not, members should provide encouragement and support to each other. All members should help the group work toward its goal.

Teacher's roles

When the group work begins, the teacher's role changes from a planner and decision maker to that of an interactant. Spending time with each group, the teacher observes and monitors the progress of cooperative learning. When necessary, teachers can clarify instructions or act as a consultant.

Whichever role the teacher assumes, care should be taken to direct students to answer questions and solve problems as a group.

When monitoring, it is usually more beneficial to focus on a few behaviors at a time. Observation sheets and checklists can be used to note positive and negative behaviors.

After the activity, the teacher's role becomes one of a guide in the debriefing process. Finally, the teacher acts as an evaluator.

Guide processing (debriefing)

Processing is a means of wrapping up the day's lesson. During processing, students discuss what they have learned and how the group has interacted. It provides an opportunity for students to look at themselves individually as well as a member of a group. Because this step is a major difference between traditional groups and cooperative learning, it should not be omitted.

Procedures for debriefing may vary. Students can verbalize how they think they performed in the group. Students can express their perceptions in written form on processing sheets that have phrases or questions such as "Our group did well...". Teacher observation sheets are another resource for discussion. Processing can take place within the group and/or as a whole class.

Evaluate the lesson

The evaluation method was planned before the lesson began and can now be implemented.

Now is the time for the teacher to think about the effectiveness of the lesson and consider any changes needed for the future.

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DEVELOPING INSTRUCTIONAL STRATEGIES FOR TEACHING FRACTIONS

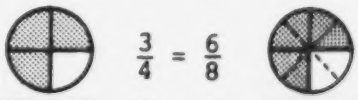
5.06.01 EQUIVALENT FRACTIONS

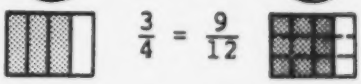
(2 lessons)

PERFORMANCE OBJECTIVES

The student will be able to...

5.06.01.01A use the equal multiplication property to generate equivalent fractions;
e.g.,

$\frac{3 \times 2}{4 \times 2} = \frac{6}{8}$ therefore $\frac{3}{4} = \frac{6}{8}$ 

$\frac{3 \times 3}{4 \times 3} = \frac{9}{12}$ therefore $\frac{3}{4} = \frac{9}{12}$ 

(This corresponds to cutting each piece into the same number of equal parts.)

5.06.01.02A rename fractions as simpler equivalent fractions in lower terms by dividing the numerator and the denominator by the same number;
e.g.,

$\frac{6 \div 3}{9 \div 3} = \frac{2}{3}$ therefore $\frac{6}{9} = \frac{2}{3}$

$\frac{4 \div 2}{8 \div 2} = \frac{2}{4}$ therefore $\frac{4}{8} = \frac{2}{4}$

$\frac{4 \div 4}{8 \div 4} = \frac{1}{2}$ therefore $\frac{4}{8} = \frac{1}{2}$

(This corresponds to bundling together fractional pieces into sets of equal size.)

5.06.01.03B express fractions in simplest form (lowest terms);
e.g.,

$$\frac{6}{12} = \frac{1}{2} \qquad \frac{30}{40} = \frac{3}{4}$$

5.06.01.04B arrange a set of fractions from the least to the greatest.
e.g.,

$$\frac{3}{4}, \frac{1}{5}, \frac{1}{2}$$

$$\frac{3 \times 5}{4 \times 5} = \frac{15}{20}$$

$$\frac{1 \times 4}{5 \times 4} = \frac{4}{20}$$

$$\frac{1 \times 10}{2 \times 10} = \frac{10}{20}$$

$$\frac{4}{20} < \frac{10}{20} < \frac{15}{20}$$

Therefore, $\frac{1}{5} < \frac{1}{2} < \frac{3}{4}$.

PROBLEM SOLVING SKILLS:

-to make a useful diagram
-to create an analogy
-to adopt a different point of view

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VOCABULARY

numerator
denominator
equivalent
lower terms
simplest form
lowest terms

PREVIOUS UNDERSTANDINGS:

- concept of fraction
- recognizing that a fraction can have many names

GETTING STARTED:

Materials:
- demonstration set of fractional parts of a circle (square, rectangle) representing halves, thirds, fourths, sixths, and eighths

Display the two halves. Ask a student to put them together to form the whole circle (square, rectangle). Say, "What is each part called?" (Each part is called $\frac{1}{2}$.) Continue this procedure to form the circles (squares, rectangles) made of thirds, fourths, sixths, and eighths.

e.g.,

Ask a student to remove $\frac{1}{2}$ of the circle which is divided into halves. Say, "What part of the circle is still displayed? ($\frac{1}{2}$ of the circle is still displayed.) Look at the circle divided into fourths. How many fourths do you think we will use to cover this $\frac{1}{2}$?" (We will use 2 fourths.) Ask a student to take $\frac{2}{4}$ of the circle divided into fourths and to place them on the piece representing $\frac{1}{2}$ of the circle. Say, "Did the $\frac{2}{4}$ completely cover the $\frac{1}{2}$? (Yes.) What can we say about $\frac{1}{2}$ of this circle and $\frac{2}{4}$ of this circle?" ($\frac{1}{2}$ of the circle and $\frac{2}{4}$ of the

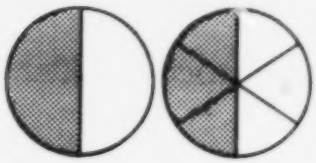
GETTING STARTED: (cont'd)

circle are the same amount.) On the chalkboard write: $\frac{1}{2} = \frac{2}{4}$. Say, "$\frac{2}{4}$ is the same amount as $\frac{1}{2}$. We can say that $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent or equal in value."

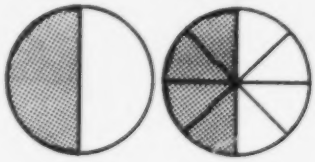
Repeat this procedure to recognize other equivalent fractions pictorially.

e.g.,

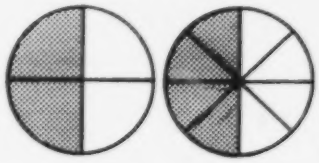
$\frac{1}{2} = \frac{3}{6}$



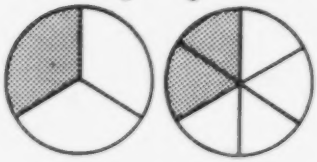
$\frac{1}{2} = \frac{4}{8}$



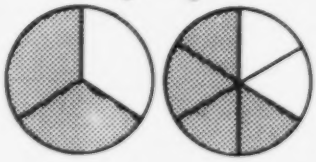
$\frac{2}{4} = \frac{4}{8}$



$\frac{1}{3} = \frac{2}{6}$



$\frac{2}{3} = \frac{4}{6}$



5.06.01

5.06.01.01 use the equal multiplication property to generate equivalent fractions

STRATEGY 1:

Materials:
- demonstration set of fractional parts of a circle (square, rectangle) representing halves, thirds, fourths, sixths, and eighths

Display the fractional parts as in the Getting Started. Say, "What are some other fractions we know that are equal to $\frac{1}{2}$?" ($\frac{2}{4}$, $\frac{3}{6}$, and $\frac{4}{8}$ are other names for this amount.) Ask several students to demonstrate each equivalence using the fractional parts. On the chalkboard write the following:

$$\frac{1}{2} = \frac{2}{4}$$
$$\frac{1}{2} = \frac{3}{6}$$
$$\frac{1}{2} = \frac{4}{8}$$

Say, "Today we will learn another way to find equivalent fractions. What do we mean by equivalent fractions?" (Equivalent fractions are fractions that are equal in value.) Point to $\frac{1}{2} = \frac{2}{4}$ on the chalkboard. Say, "Look at both numerators. How many times greater is 2 than 1? (2 is two times greater than 1.) Look at the denominators. How many times greater is 4 than 2?" (4 is two times greater than 2.) On the chalkboard write: $\frac{1}{2} \frac{x2}{x2} = \frac{2}{4}$. Ask the students to write it in their notebooks.

Say, "We multiplied the numerator and the denominator by 2. We know that $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent because we have seen it demonstrated with fractional parts. What is another way to find an equivalent of any fraction? (Another way to find an equivalent of any fraction is by multiplying the numerator and the denominator by the same number.) In this case we used 2 as the multiplier, but we could have used any other number. Let's start with $\frac{1}{2}$ again and multiply the numerator and denominator by 3." Ask the students to do the multiplication in their

STRATEGY 1: (cont'd)

notebooks as one student works at the chalkboard.
e.g.,

$$\frac{1}{2}\ \frac{x3}{x3} = \frac{3}{6}$$

Say, "What other equivalent fraction did we find for $\frac{1}{2}$? (We found that $\frac{3}{6}$ is equivalent to $\frac{1}{2}$.) Can we prove this by using the fractional parts?" (yes) Ask a student to demonstrate this equivalence using the fractional parts.
e.g.,

$\frac{1}{2} = \frac{3}{6}$

Repeat this procedure to find equivalents for various fractions. Find several new names for a given fractional amount.
e.g.,

$$\frac{1}{3}\ \frac{x2}{x2} = \frac{2}{6} \qquad \frac{1}{3}\ \frac{x3}{x3} = \frac{3}{9} \qquad \frac{1}{3}\ \frac{x4}{x4} = \frac{4}{12}$$

$$\frac{2}{5}\ \frac{x2}{x2} = \frac{4}{10} \qquad \frac{2}{5}\ \frac{x3}{x3} = \frac{6}{15} \qquad \frac{2}{5}\ \frac{x4}{x4} = \frac{8}{20}$$

Where possible, the students may prove the equivalence using the fractional parts.

On the chalkboard write: $\frac{1}{5} = \frac{\square}{10}$. Say, "How can we find the missing numerator? (The equal symbol shows that the two fractions are equivalent. If we know by which number the denominator 5 was multiplied to get the denominator 10, then we can multiply the numerator by that same number.) By what number was the denominator 5 multiplied to get 10? (5 was multiplied by 2 to get 10.) By what number must we multiply the numerator?" (To be sure the fractions are equivalent, we must also multiply the numerator by 2.)
Ask a student to write this on the chalkboard as the other students write it in their notebooks.
e.g.,

$$\frac{1}{5}\ \frac{x2}{x2} = \frac{2}{10}$$

Say, "A fraction equivalent to $\frac{1}{5}$, but named as tenths, is $\frac{2}{10}$. Because the new numerator and the new denominator are larger numbers, we say

STRATEGY 1: (cont'd)

this equivalent fraction is in higher terms." Repeat this procedure using other fractions.

e.g.,

$\frac{1}{3} = \frac{\square}{12}$ $\frac{2}{7} = \frac{6}{\square}$

STRATEGY 2:

Materials:
- 2 blank sheets of paper for each student
- crayons

Distribute a sheet of paper to each student. Say, "Fold your paper in half. Now open it. How many equal parts are there? (There are 2 equal parts.) What can we call each part? (We can call each part $\frac{1}{2}$ of the whole paper.) Using your crayon, shade $\frac{1}{2}$ of the whole paper." After the students have finished shading say, "Hold your paper so that only the shaded half is showing."

e.g.,

Say, "Now fold your paper in half again."

e.g.,

Say, "Now open your paper completely. Now how many equal parts are there? (There are 4 equal parts.) What fractional part of the page is shaded? ($\frac{2}{4}$ is shaded.) Did we change the amount of shading? (no) The shaded amount remained the same but we have named it in two different ways." On the chalkboard write $\frac{1}{2} = \frac{2}{4}$. Repeat this procedure by folding another piece of paper into thirds, shading $\frac{2}{3}$ of the paper, and then folding it in half to show that $\frac{2}{3} = \frac{4}{6}$. Write this

STRATEGY 2: (cont'd)

equivalence on the chalkboard.

Point to $\frac{1}{2} = \frac{2}{4}$ on the chalkboard and say, "One half and two fourths are equivalent. We proved that with paper folding. Now we will see another way to find equivalents for any fraction." Write the following on the chalkboard: $\frac{1}{2} \frac{x\square}{x\square} = \frac{2}{4}$. Say, "Look at the numerators. One times what number equals 2?" (1 times 2 equals 2.) In the box in the numerator, write 2. Say, "Look at the denominators. Two times what number equals 4?" (Two times two equals 4.) In the box in the denominator, write 2. Say, "What do you think happens when we multiply the numerator and denominator by the same number?" (It always produces a fraction that is equivalent to the fraction with which we started.) Ask several students to find other fractions that are equivalent to $\frac{1}{2}$ by multiplying the numerator and the denominator by 3, by 4, and by 5.

e.g.,

$$\frac{1}{2} \frac{x3}{x3} = \frac{3}{6} \qquad \frac{1}{2} = \frac{3}{6}$$

$$\frac{1}{2} \frac{x4}{x4} = \frac{4}{8} \qquad \frac{1}{2} = \frac{4}{8}$$

$$\frac{1}{2} \frac{x5}{x5} = \frac{5}{10} \qquad \frac{1}{2} = \frac{5}{10}$$

Say, "All the fractions we generated were in higher terms than the original fractions. How could we create other fractions equivalent to $\frac{1}{2}$?" (We could multiply the numerator and the denominator by 4, 5, 6, ...)

On the chalkboard, point to $\frac{2}{3} = \frac{4}{6}$, which the students discovered through paper folding. Say, "What number could we use to multiply the numerator and the denominator of $\frac{2}{3}$ to get the equivalent fraction $\frac{4}{6}$?" (We could multiply the numerator and the denominator by 2.)

On the chalkboard write $\frac{2}{3} \frac{x2}{x2} = \frac{4}{6}$. Ask the students to write this in their notebooks and to generate other fractions equivalent to $\frac{2}{3}$ by multiplying the numerator and the denominator by the same number.

STRATEGY 2: (cont'd)

e.g.,

$$\frac{2}{3}\ \frac{x4}{x4} = \frac{8}{12} \qquad \frac{2}{3} = \frac{8}{12}$$

$$\frac{2}{3}\ \frac{x6}{x6} = \frac{12}{18} \qquad \frac{2}{3} = \frac{12}{18}$$

$$\frac{2}{3}\ \frac{x10}{x10} = \frac{20}{30} \qquad \frac{2}{3} = \frac{20}{30}$$

On the chalkboard write $\frac{2}{5} = \frac{\square}{15}$. Say, "How can we find the numerator for a fraction with a denominator of 15 that will be equivalent to $\frac{2}{5}$?" (Since the denominator 5 was multiplied by 3 to get the denominator 15, then we should also multiply the numerator by 3.) Ask a student to demonstrate this on the chalkboard as the other students write it in their notebooks.

e.g.,

$$\frac{2}{5}\ \frac{x3}{x3} = \frac{6}{15} \qquad \frac{2}{5} = \frac{\boxed{6}}{15}$$

Repeat this procedure using other examples.

5.06.01.02 rename fractions as simpler equivalent fractions in lower terms by dividing the numerator and the denominator by the same number

STRATEGY 1:

On the chalkboard, write the fractions $\frac{2}{4}$, $\frac{5}{10}$, $\frac{1}{2}$, $\frac{6}{12}$, $\frac{30}{60}$. Ask, "How are these fractions the same? How are they different? (They all have the same value, each fraction equals one half; the numerators and denominators vary, the whole is cut into a different number of equal parts each time.) Which of these fractions is the easiest to visualize?" ($\frac{1}{2}$, because the whole is cut into the least number of equal pieces)

Adapt Strategy 1, P.O. 5.06.01.01, beginning with fractions in higher terms, e.g., $\frac{3}{6}$, and dividing the numerator and the denominator by the same number to create a lower-terms equivalent. Ask, "When we changed fractions into higher-terms equivalents, we saw that we could think about this as cutting each piece into many parts. How shall we think about putting a fraction into lower terms?" (Students may suggest grouping or bundling or pasting parts together.)

STRATEGY 2:

Adapt Strategy 2, P.O. 5.06.01.01, by first dividing the paper into fourths and shading two-fourths. Then ask the students to use a dark crayon to trace over the center line of the fold, and show the renaming of $\frac{2}{4}$ as $\frac{1}{2}$. Students should discover that this renaming process is related to dividing the numerator and the denominator by the same number.

5.06.01.03 express fractions in simplest form (lowest terms)

STRATEGY 1:

Say, "We have learned that different fractions can be worth the same amount. What do we call fractions that have the same value? (We call them equivalent fractions.) How did we generate equivalent fractions in higher terms for a given one? (We multiplied the numerator and the denominator by the same number.) Is there a limit to the number of higher term equivalents we could create? Explain. (No, because there is no limit to the numbers we can use as multipliers.) How did we find equivalent fractions in lower terms? (We divided the numerator and the denominator by the same number.) Is there a limit to the numbers we can use as divisors? Explain. (There is a limit because the divisor must be a factor of both the numerator and the denominator. Some numbers will be divisors, but not all. We can only use divisors, not just any number we choose.) Now we will learn to express fractions in simplest form or lowest terms."

On the chalkboard write $\frac{12}{20}$. Say, "What divisor can we use that is a factor of both the numerator and the denominator?" (2 is a factor of 12 and also a factor of 20. Therefore, we can use 2 as a divisor.) Four would be a better choice, but the students will probably not realize it at this point. Ask the students to divide the numerator and the denominator by 2 to find an equivalent fraction in lower terms as one student works at the chalkboard.

e.g.,

$$\frac{12}{20} \begin{matrix}+2\\+2\end{matrix} = \frac{6}{10} \qquad \frac{12}{20} = \frac{6}{10}$$

Say, "What fraction in lower terms is equivalent to $\frac{12}{20}$? ($\frac{6}{10}$ is equivalent to $\frac{12}{20}$.) Now look at $\frac{6}{10}$. Is there a divisor we can use

STRATEGY 1: (cont'd)

that is a factor of 6 and 10?" (2 is a factor of 6 and 10.) Ask the students to reduce $\frac{6}{10}$ by dividing the numerator and the denominator by 2 as a student works at the chalkboard.

e.g.,

$$\frac{6}{10} \frac{+2}{+2} = \frac{3}{5} \qquad \frac{12}{20} = \frac{6}{10} = \frac{3}{5}$$

Say, "$\frac{3}{5}$ is equivalent to $\frac{6}{10}$ and also to $\frac{12}{20}$. Can we reduce $\frac{3}{5}$ to an equivalent fraction in lower terms? Explain. (No, there is no number other than 1 which is a factor of 3 and 5.) Why can't we use 1 as the divisor? (We can, but dividing by 1 will not reduce the fraction further.) Since we can't reduce further, we can say that $\frac{12}{20}$ reduced to <u>lowest</u> terms is $\frac{3}{5}$. How many steps did this take? (We did this in two steps.) How could we have done this in one step? Explain." (We could have divided the numerator and the denominator of $\frac{12}{20}$ by 4.) Ask a student to show this on the chalkboard as the other students work in their notebooks.

e.g.,

$$\frac{12}{20} \frac{+4}{+4} = \frac{3}{5}$$

Repeat this procedure to rename other fractions in simplest form (lowest terms).

5.06.01.04 arrange a set of fractions from the least to the greatest

STRATEGY 1:

Materials:

- fractional parts for halves, thirds, fourths, fifths, eighths

Write the following on the chalkboard:

$\frac{1}{3} \bigcirc \frac{2}{3}$ $\qquad$ $\frac{1}{4} \bigcirc \frac{3}{4}$

$\frac{4}{5} \bigcirc \frac{2}{5}$ $\qquad$ $\frac{3}{8} \bigcirc \frac{7}{8}$

Say, "Look at the first pair of fractions. How does $\frac{1}{3}$ compare to $\frac{2}{3}$?

STRATEGY 1: (cont'd)

($\frac{1}{3}$ is less than $\frac{2}{3}$.) How can you be sure?" (Since both fractions are expressed as thirds, the pieces are the same size. Having 1 of those pieces is less than having 2 of those pieces.) Ask a student to write the correct symbol between the fractions.

e.g.,

$$\frac{1}{3} \textcircled{<} \frac{2}{3}$$

Repeat this procedure with the remaining fractions. Ask the students to verify their answers by using the fractional parts.

e.g.,

$\frac{1}{3} \textcircled{<} \frac{2}{3}$

Say, "It is easy to compare fractions with the same denominator because we are talking about pieces that are the same size."

Write the following on the chalkboard:

$$\frac{3}{4} \bigcirc \frac{2}{5}$$

Say, "How is this pair of fractions different from the other pairs of fractions we compared? (These fractions have different denominators as well as different numerators.) What have you learned that will make comparing these fractions easier? (We have learned to find equivalent fractions which have new denominators.) How can we find equivalent fractions? (We can multiply the numerator and the denominator by the same number.) In this problem, it will be helpful to replace these fractions with others which are equivalent to them, but which have a new denominator. The new denominator, then, must be a multiple of both denominators 4 and 5. What number is a multiple of 4 and 5? (20 is a multiple of 4 and 5.) Then 20 will be the denominator of both equivalent fractions."

Write the following on the chalkboard:

$$\frac{3}{4} = \frac{\square}{20}$$

$$\frac{2}{5} = \frac{\square}{20}$$

STRATEGY 1: (cont'd)

Say, "By what number will we multiply the numerator and the denominator of $\frac{3}{4}$ to get an equivalent fraction expressed as twentieths?" (We will multiply by 5.) Ask a student to show the computation at the chalkboard as the other students write it in their notebooks.

e.g.,

$$\frac{3}{4} \frac{x5}{x5} = \frac{15}{20}$$

Repeat this procedure to rename $\frac{2}{5}$ as $\frac{8}{20}$. Say, "These new fractions, $\frac{15}{20}$ and $\frac{8}{20}$, are equivalent to our original fractions, $\frac{3}{4}$ and $\frac{2}{5}$. Why are they easier to compare now?" (They have the same denominators.) On the chalkboard write the following:

$$\frac{3}{4} \bigcirc \frac{2}{5}$$

$$\frac{15}{20} \bigcirc \frac{8}{20}$$

Say, "How does $\frac{15}{20}$ compare to $\frac{8}{20}$? ($\frac{15}{20}$ is greater than $\frac{8}{20}$.) Then how does $\frac{3}{4}$ compare to $\frac{2}{5}$?" ($\frac{3}{4}$ is greater than $\frac{2}{5}$.) Ask a student to place the greater than symbol in both expressions on the chalkboard as the other students do the same in their notebooks.

e.g.,

$$\frac{3}{4} \textcircled{>} \frac{2}{5}$$

$$\frac{15}{20} \textcircled{>} \frac{8}{20}$$

Ask a student to verify this by using the fractional parts.

e.g,

$$\frac{3}{4} \textcircled{>} \frac{2}{5}$$

Say, "How did we make it easy to compare fractions with different denominators? (We renamed the fractions as equivalents with the same denominator.) It was easier to compare these amounts when we changed our point of view and saw the fractions as twentieths instead of fourths and fifths. How did we prove this works?" (We checked by

STRATEGY 1: (cont'd)

comparing the fractional parts.)

Repeat this procedure comparing and ordering three fractions.

e.g.,

$$\frac{2}{3},\ \frac{1}{2},\ \frac{3}{4}$$

$$\frac{2}{3}\ \frac{x4}{x4} = \frac{8}{12}$$

$$\frac{1}{2}\ \frac{x6}{x6} = \frac{6}{12}$$

$$\frac{3}{4}\ \frac{x3}{x3} = \frac{9}{12}$$

In increasing order the equivalents are: $\frac{6}{12}, \frac{8}{12}, \frac{9}{12}$.

Therefore, in increasing order the original fractions are: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

Students should be encouraged to create similar problems (with solutions) to challenge each other.

P.O. 5.06.01.01-.02

NAME: _____ CLASS: _____ DATE: _____

I. Find equivalent fractions in higher terms.

ⓐ $\frac{1}{2}$ $\frac{x3}{x3}$ = ☐ $\frac{1}{2}$ = ☐

ⓑ $\frac{3}{5}$ $\frac{x2}{x2}$ = ☐ $\frac{3}{5}$ = ☐

ⓒ $\frac{2}{3}$ $\frac{x5}{x5}$ = ☐ $\frac{2}{3}$ = ☐

ⓓ $\frac{4}{7}$ $\frac{x2}{x2}$ = ☐ $\frac{4}{7}$ = ☐

ⓔ $\frac{3}{4}$ $\frac{x4}{x4}$ = ☐ $\frac{3}{4}$ = ☐

II. Write the correct numerator or denominator to complete each pair of equivalent fractions.

ⓐ $\frac{1}{3} = \frac{\square}{6}$

ⓑ $\frac{4}{5} = \frac{8}{\square}$

ⓒ $\frac{1}{4} = \frac{3}{\square}$

ⓓ $\frac{3}{4} = \frac{\square}{16}$

ⓔ $\frac{2}{8} = \frac{6}{\square}$

III. Find equivalent fractions in lower terms.

ⓐ $\frac{6}{9}$ $\frac{+3}{+3}$ = ☐ $\frac{6}{9}$ = ☐

ⓑ $\frac{10}{15}$ $\frac{+5}{+5}$ = ☐ $\frac{10}{15}$ = ☐

ⓒ $\frac{12}{24}$ $\frac{+2}{+2}$ = ☐ $\frac{12}{24}$ = ☐

ⓓ $\frac{5}{10}$ $\frac{+5}{+5}$ = ☐ $\frac{5}{10}$ = ☐

ⓔ $\frac{4}{6}$ $\frac{+2}{+2}$ = ☐ $\frac{4}{6}$ = ☐

IV. Write the correct numerator or denominator to complete each pair of equivalent fractions.

ⓐ $\frac{5}{10} = \frac{\square}{2}$

ⓑ $\frac{4}{10} = \frac{2}{\square}$

ⓒ $\frac{3}{6} = \frac{1}{\square}$

ⓓ $\frac{8}{12} = \frac{\square}{2}$

ⓔ $\frac{10}{20} = \frac{2}{\square}$

P.O. 5.06.01.03-.04

NAME: _____ CLASS: _____ DATE: _____

A. Express the following fractions in simplest form.

① $\frac{10}{20} = \frac{1}{2}$ ② $\frac{5}{10} =$ ③ $\frac{2}{6} =$ ④ $\frac{4}{12} =$

⑤ $\frac{2}{8} =$ ⑥ $\frac{2}{10} =$ ⑦ $\frac{4}{8} =$ ⑧ $\frac{5}{15} =$

⑨ $\frac{6}{18} =$ ⑩ $\frac{8}{10} =$ ⑪ $\frac{6}{14} =$ ⑫ $\frac{4}{10} =$

⑬ $\frac{9}{12} =$ ⑭ $\frac{12}{16} =$ ⑮ $\frac{4}{20} =$ ⑯ $\frac{5}{25} =$

B. Arrange the following sets of fractions in order from the greatest to the least.

① $\left\{\frac{2}{3}, \frac{1}{2}, \frac{1}{4}\right\}$ ② $\left\{\frac{5}{6}, \frac{1}{8}, \frac{3}{5}\right\}$ ③ $\left\{\frac{3}{4}, \frac{1}{5}, \frac{3}{10}\right\}$

__ __ __ __ __ __ __ __ __

Do you remember?

Write in expanded notation.

340,821 = _____

513,497 = _____

840,502 = _____

706,216 = _____

Unit 7 **5.06.02 IMPROPER FRACTIONS AND MIXED NUMERALS**

(2 lessons)

PERFORMANCE OBJECTIVES

The student will be able to...

5.06.02.01A change an improper fraction to a mixed numeral or a whole number;
e.g.,

$$\frac{4}{3} = 1\frac{1}{3}$$

$$\frac{5}{5} = 1$$

$$\frac{8}{4} = 2$$

5.06.02.02A change a mixed numeral or a whole number to an improper fraction.
e.g.,

$$2\frac{1}{3} = \frac{7}{3}$$

$$4 = \frac{8}{2}$$

PROBLEM SOLVING SKILLS:

-to state the goals of a problem clearly
-to compare and contrast

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VOCABULARY

proper fraction
improper fraction
numerator
denominator

PREVIOUS UNDERSTANDINGS:

- common denominator
- identification and definitions of improper fractions and mixed numerals

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GETTING STARTED:

Materials:

- flannel board, flannel fractional parts, flannel letters

On the flannel board, display the following:

Say, "The fractional parts displayed on this flannel board demonstrate proper fractions, improper fractions, and mixed numerals." Ask the students to identify each of the above figures and explain why such identification is appropriate.

e.g.,

'A' shows the mixed numeral $1\frac{1}{3}$. $1\frac{1}{3}$ is a mixed numeral because it shows 1 whole circle and a fraction of a circle. 'A' also shows the improper fraction $\frac{4}{3}$. $\frac{4}{3}$ is an improper fraction because the numerator is greater than the denominator. Notice that the whole is divided into 3 equal parts, each being $\frac{1}{3}$. Therefore,

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}.$$

Repeat this procedure with illustrations B through F.

5.06.02

5.06.02.01 change an improper fraction to a mixed numeral or a whole number

STRATEGY 1:

Materials:
- flannel board, flannel fractional parts

Display fractional parts as in the Getting Started activity for this unit.

Say, "We have seen how a given figure, such as Figure A, can be associated with both a mixed numeral and an improper fraction. Which other figures can be seen in two ways?" (Figures C, D and E.) Ask several students to use the fractional parts to demonstrate other mixed numerals and improper fractions. On the flannel board display seven halves.

e.g.,

Say, "What improper fraction describes these pieces? Explain." (This shows 7 halves. Each piece is $\frac{1}{2}$ of a circle. 7 halves are greater than 1 whole.) Ask a student to use these pieces to make as many whole circles as possible.

e.g.,

Say, "What mixed numeral describes these pieces? Explain." (This shows $3\frac{1}{2}$. We were able to form 3 whole circles and $\frac{1}{2}$ circle from the 7 half circles.) Since we used all the pieces to change $\frac{7}{2}$ to $3\frac{1}{2}$, we can say $\frac{7}{2} = 3\frac{1}{2}$." On the chalkboard write: $\frac{7}{2} = 3\frac{1}{2}$. Say, "If we want to change an improper fraction to a mixed numeral, it is not necessary to use fractional pieces. What operation is suggested when we see the bar in the fraction $\frac{7}{2}$? Explain. (Division is suggested. $\frac{7}{2}$ also means 7 divided by 2.) How much is 7 divided by 2? (7 divided by 2 equals 3 with a remainder of 1.) To write the mixed numeral that is

STRATEGY 1: (cont'd)

equivalent to $\frac{7}{2}$, the 3 is written as the whole number and the remainder is written as a fraction with the divisor, 2, as the denominator." On the chalkboard write: $\frac{7}{2} = 2\overline{)7} = 3\frac{1}{2}$.

On the flannel board display 4 halves.
e.g.,

Say, "What improper fraction describes these fractional pieces? Explain." (This shows 4 halves. Each piece is $\frac{1}{2}$ of a circle.) Ask a student to write the fraction, $\frac{4}{2}$, on the chalkboard. Ask another student to use the 4 half circles to make as many whole circles as possible.
e.g.,

Say, "How many circles did we make?" (We made 2 whole circles.) On the chalkboard write $\frac{4}{2} = 2$. Say, "How could we determine that $\frac{4}{2} = 2$ without using the fractional pieces? (We could divide the numerator by the denominator. 4 divided by 2 equals 2.) Is there a remainder? (No.) Therefore $\frac{4}{2}$ can be expressed as a whole number. What have we learned about the ways in which improper fractions may be expressed?" (Improper fractions may be expressed as mixed numerals and whole numbers.)

Repeat this procedure to change other improper fractions to mixed numbers or whole numbers.
e.g.,

$\frac{5}{3} = 1\frac{2}{3}$ $\quad \frac{9}{5} = 1\frac{4}{5}$ $\quad \frac{9}{2} = 4\frac{1}{2}$

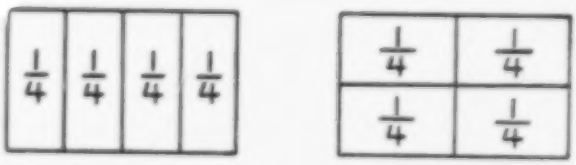
$\frac{8}{4} = 2$ $\quad \frac{10}{6} = 1\frac{4}{6}$ $\quad \frac{10}{5} = 2$

5.06.02.02 change a mixed numeral or a whole number to an improper fraction

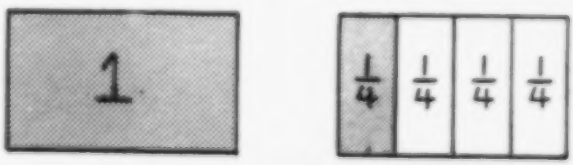
STRATEGY 1:

Materials:
- small note paper for folding; 2 sheets per student or group of students
- crayons

On the chalkboard write the mixed numeral $1\frac{1}{4}$. Say, "What kind of number is $1\frac{1}{4}$? Explain." ($1\frac{1}{4}$ is a mixed numeral because it is made up of a whole number and a fraction.) Distribute crayons and 2 sheets of note paper to each student or group of students. Say, "We can use this paper to demonstrate the mixed numeral $1\frac{1}{4}$. Place 1 whole sheet on your desk and write 1 on it, e.g., [1]. Fold the other sheet into 4 equal parts, e.g., [| | |] or [— —]. What can we call each part? (one fourth or one quarter) Write the fraction $\frac{1}{4}$ in each part."
e.g.,



Say, "Use a crayon to shade the unfolded paper and $\frac{1}{4}$ of the folded paper. What is now shaded? (1 whole and one fourth are now shaded.) We know that an improper fraction can be expressed as a mixed numeral or a whole number. Therefore, we can also express a mixed numeral as an improper fraction. To express $1\frac{1}{4}$ as an improper fraction, what would its denominator be? Explain." (The denominator would be 4 because 4 is the denominator of one-fourth.) Ask the students to fold their unfolded paper into fourths. Say, "How many fourths in all are now shaded?" ($\frac{5}{4}$ are now shaded.)
e.g.,



STRATEGY 1: (cont'd)

Say, "We can say that $1\frac{1}{4}$ equals $\frac{5}{4}$." On the chalkboard write: $1\frac{1}{4} = \frac{5}{4}$. Say, "Just as we do not need pictures to change an improper fraction to a mixed numeral, we do not need pictures to change a mixed numeral to an improper fraction. What operation did we use to change an improper fraction to a mixed numeral? (division) What operation do you think we should use to change a mixed numeral to an improper fraction? Explain." (We should use multiplication because it is the inverse operation of division.)

Ask the students to review the steps for changing a mixed numeral to an improper fraction as a student records these steps on the chalkboard.

e.g., **How to Change a Mixed Numeral to an Improper Fraction**

$1\frac{1}{4}$

1. Multiply the denominator and the whole number.
   $4 \times 1 = 4$
2. Add the numerator to this product. The sum is the new numerator. $4 + 1 = 5$
3. Use the denominator from the fraction for the improper fraction. $\frac{5}{4}$
4. Therefore, $1\frac{1}{4} = \frac{5}{4}$.

Ask the students to record these steps in their notebooks. Provide further practice by asking several students to write mixed numerals on the chalkboard which will be changed to improper fractions.

Say, "What would we be more likely to use in problem solving, mixed numerals or improper fractions? Explain." (We would be more likely to use mixed numerals. When measuring various amounts they are usually expressed in whole numbers or mixed numerals, e.g., gallons of gasoline needed to fill a tank, yards of fabric needed to sew a dress, the length or width of a room. This gives us a better sense of the size of the amount.)

To review, ask, "When $\frac{9}{4}$ is changed to $2\frac{1}{4}$, what has been done? (An improper fraction has been changed to a mixed numeral.) Give an example of a mixed numeral being changed to an improper fraction and explain the procedure." (Answers will vary.)

P.O. 5.06.02.01-.02

NAME: _____ CLASS: _____ DATE: _____

A. Name each shaded amount as a mixed numeral or a whole number and as an improper fraction.

1.

$3\frac{1}{2}$ or $\frac{7}{2}$

2.

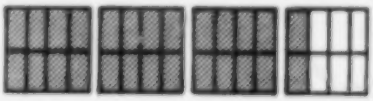
_____ or _____

3.

_____ or _____

4.

_____ or _____

5. 

_____ or _____

6.

_____ or _____

B. Rename each mixed numeral as an improper fraction.

1. $4\frac{1}{5}$ = _____
2. $3\frac{2}{3}$ = _____
3. $7\frac{1}{2}$ = _____
4. $5\frac{1}{8}$ = _____
5. $6\frac{3}{8}$ = _____
6. $4\frac{5}{6}$ = _____
7. $2\frac{7}{8}$ = _____
8. $1\frac{1}{3}$ = _____
9. $9\frac{1}{3}$ = _____
10. $3\frac{1}{4}$ = _____
11. $5\frac{1}{2}$ = _____
12. $4\frac{1}{5}$ = _____

C. Rename each improper fraction as a mixed numeral or whole number.

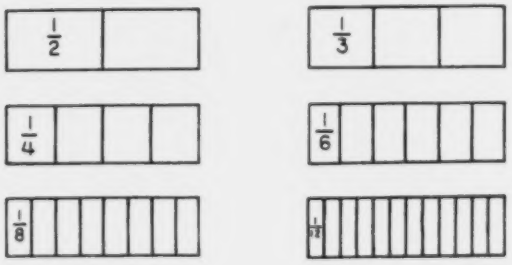
1. $\frac{5}{2}$ = _____
2. $\frac{10}{3}$ = _____
3. $\frac{12}{5}$ = _____
4. $\frac{17}{3}$ = _____
5. $\frac{9}{2}$ = _____
6. $\frac{27}{5}$ = _____
7. $\frac{20}{4}$ = _____
8. $\frac{31}{7}$ = _____
9. $\frac{7}{3}$ = _____
10. $\frac{12}{4}$ = _____
11. $\frac{24}{9}$ = _____
12. $\frac{28}{5}$ = _____

SHARPENING YOUR SKILLS

Units 5, 6, 7

Name: _____ **Class:** _____ **Date:** _____

1. Use the pictures below to help you complete each equivalent fraction.



$\frac{1}{2} = \frac{\square}{8}$ $\quad \frac{\square}{3} = \frac{4}{6}$

$\frac{3}{4} = \frac{6}{\square}$ $\quad \frac{8}{12} = \frac{\square}{3}$

2. Find the sum or difference.

$\frac{5}{8} + \frac{1}{8} = \square$

$$\begin{array}{r} \frac{2}{9} \\ +\ \frac{3}{9} \\ \hline \square \end{array} \qquad \begin{array}{r} \frac{8}{10} \\ -\ \frac{1}{10} \\ \hline \square \end{array}$$

3. Find an equivalent fraction.

$\frac{3}{4} = $ _____

(a) $\frac{3}{8}$ (b) $\frac{6}{12}$ (c) $\frac{9}{12}$ (d) $\frac{6}{7}$

4. Which fraction expresses $\frac{5}{10}$ in lowest terms?

(e) $\frac{1}{10}$ (f) $\frac{1}{2}$ (g) $\frac{2}{7}$ (h) $\frac{1}{5}$

5. By what number would we divide to rename $\frac{6}{9}$ in lowest terms?

_____

(a) 6 (b) 9 (c) 2 (d) 3

6. Arrange each set of fractions from the least to the greatest.

$\frac{5}{8}, \frac{3}{4}, \frac{1}{2}$ $\quad \square > \square > \square$

$\frac{2}{3}, \frac{2}{5}, \frac{1}{4}$ $\quad \square > \square > \square$

7. Rename each improper fraction as a mixed numeral or whole number.

$\frac{9}{2} = \square$ $\quad \frac{16}{4} = \square$

$\frac{8}{3} = \square$ $\quad \frac{18}{5} = \square$

8. Rename each mixed numeral as an improper fraction.

$2\frac{1}{8} = \square$ $\quad 4\frac{2}{3} = \square$ $\quad 1\frac{1}{10} = \square$

Unit 8 **5.06.03 MULTIPLICATION WITH FRACTIONS**

(4-5 lessons)

PERFORMANCE OBJECTIVES

The student will be able to...

5.06.03.01A multiply a whole number by a number expressed as a fraction and rename it in lowest (simplest) terms;

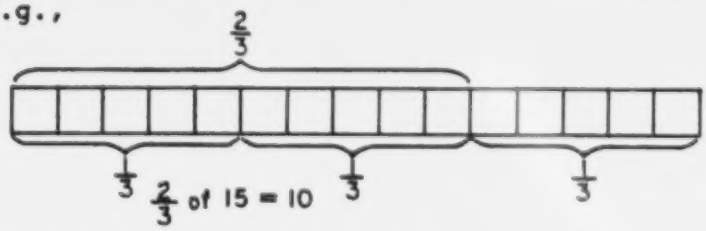
e.g.,

$12 \times \frac{1}{5} = \frac{12}{5}$ (or $2\frac{2}{5}$) $\quad 2 \times \frac{3}{7} = \frac{6}{7}$

$6 \times \frac{1}{3} = \frac{6}{3}$ (or 2) $\quad 10 \times \frac{1}{2} = \frac{10}{2}$ (or 5)

5.06.03.02A find a fractional part of a whole number by creating equal parts (given a fraction whose denominator is a divisor of the whole number);

e.g.,



*5.06.03.03A explain that the product of any unit fraction (fraction having a numerator of one) and a whole number can be interpreted as finding that fractional part of the number; and find such products when the whole number is a multiple of the denominator of the fraction;

e.g.,

$\frac{1}{2} \times 10 = \frac{1}{2}$ of $10 = 5$

$\frac{1}{2} \times 60 = \frac{1}{2}$ of $60 = 30$

$\frac{1}{4} \times 60 = \frac{1}{4}$ of $60 = 15$

$\frac{1}{3} \times 12 = \frac{1}{3}$ of $12 = 4$

$\frac{1}{5} \times 12 = \frac{1}{5}$ of $12 = \frac{12}{5}$ or $2\frac{2}{5}$

(This may be verified by using $2\frac{2}{5}$ as an addend five times.)

5.06.03.04B multiply a unit fraction by a second fraction by extending the understanding of taking a fractional part;
e.g.,

$\frac{1}{2} \times 10 = \frac{1}{2}$ of $10 = 5$

$\frac{1}{2} \times 10{,}000 = \frac{1}{2}$ of $10{,}000 = 5{,}000$

$\frac{1}{2} \times \frac{10}{17} = \frac{1}{2}$ of $\frac{10}{17} = \frac{5}{17}$

Note: It may be necessary to rename the second fraction as a higher terms equivalent whose numerator is a multiple of the first denominator.

$\frac{1}{2} \times \frac{5}{7} = \frac{1}{2}$ of $\frac{5}{7}$ (Note: $\frac{5}{7} = \frac{10}{14}$)

$= \frac{1}{2}$ of $\frac{10}{14} = \frac{5}{14}$

5.06.03.05B multiply <u>any</u> two fractions less than one by extending the understanding of taking a fractional part;
e.g.,
To find $\frac{2}{3}$ of a number, first find $\frac{1}{3}$ of the number and then double the result.

$\frac{2}{3} \times \frac{12}{25} = 2 \times \frac{1}{3}$ of $\frac{12}{25} = 2 \times \frac{4}{25} = \frac{8}{25}$

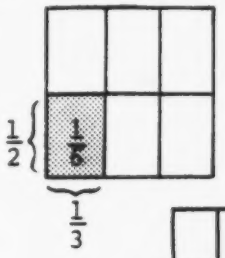
$\frac{2}{3} \times \frac{5}{8} = 2 \times \frac{1}{3}$ of $\frac{5}{8}$ (Note:) $\frac{5}{8} = \frac{15}{24}$

$= 2 \times \frac{1}{3}$ of $\frac{15}{24}$

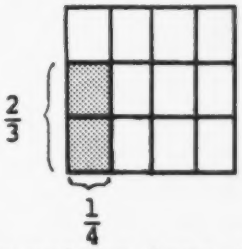
$= 2 \times \frac{5}{24} = \frac{10}{24}$ or $\frac{5}{12}$

*5.06.03.06A make a picture or other graphic illustration to demonstrate multiplication of fractions less than one;
e.g.,

$\frac{1}{3} \times \frac{1}{2} = \frac{1}{3}$ of $\frac{1}{2} = \frac{1}{6}$



$\frac{2}{3} \times \frac{1}{4} = \frac{2}{3}$ of $\frac{1}{4} = \frac{2}{12}$ or $\frac{1}{6}$



5.06.03.07B multiply with fractions having values less than 1 by taking products of numerators and denominators;
e.g.,

$\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

5.06.03.08B create and solve word problems related to skills learned in this module.
e.g.,
There are 24 students in the class. $\frac{2}{3}$ of them are present. How many students are present?

Raisins cost $2.00 per pound.
How much does $\frac{1}{2}$ lb cost?

PROBLEM SOLVING SKILLS:

-to generalize
-to state the goals of a problem clearly
-to adopt a different point of view
-to use estimation to check the reasonableness of an answer

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VOCABULARY

numerator
denominator
equivalent fraction
lowest terms
higher terms
multiple

PREVIOUS UNDERSTANDINGS:

- finding equivalent fractions in higher and lower terms

GETTING STARTED:

Materials:
- fractional parts
- fraction cards:

Set A: $\frac{9}{2}$ $3\frac{1}{3}$ $\frac{7}{2}$ $\frac{4}{8}$ $5\frac{1}{2}$ $\frac{4}{2}$ $\frac{4}{6}$ $1\frac{2}{5}$

Set B: $4\frac{1}{2}$ $\frac{10}{3}$ $3\frac{1}{2}$ $\frac{1}{2}$ $\frac{11}{2}$ 2 $\frac{2}{3}$ $\frac{7}{5}$

Display the cards from Set A. Ask the students to identify the fractions as either proper fractions, improper fractions, or mixed numerals. Shuffle the cards from Set B and place them face down on a table. Say, "For each fraction or mixed numeral you see, there is an equivalent in the set of cards on the table." Ask a student to select a card from Set B and match it to its equivalent in Set A.

e.g.,

Set A: $\frac{9}{2}$ $3\frac{1}{3}$ $\frac{7}{2}$ $\frac{4}{8}$ $5\frac{1}{2}$ $\frac{4}{2}$ $\frac{4}{6}$ $1\frac{2}{5}$

Set B: $3\frac{1}{2}$

Ask another student to verify the match by using the fractional parts.

e.g.,

$\frac{7}{2}$ or $3\frac{1}{2}$

Repeat this procedure until each match has been made and verified.

5.06.03

5.06.03.01 **multiply a whole number by a number expressed as a fraction and rename it in lowest (simplest) terms**

STRATEGY 1:

Materials:
- fractional parts

Say, "I went to the store five times and each time I bought two boxes of cookies. How can I show this with mathematics?" (five times two) On the chalkboard write: 5 x 2. Say, "What is another way of thinking about 5 times 2?" (We can think of it as adding 2 five times.) Ask a student to demonstrate this on the chalkboard, e.g., 2+2+2+2+2. Say, "How much is 5 groups of 2?" (5 groups of 2 equal 10.) Write 10 after both examples on the chalkboard.
e.g.,

$5 \times 2 = 10$ $\qquad$ $2 + 2 + 2 + 2 + 2 = 10$

On the chalkboard write: $5 \times \frac{1}{2} = \square$. Say, "What is another way of thinking about $5 \times \frac{1}{2}$? (We can think of it as adding $\frac{1}{2}$ five times.) Ask a student to demonstrate by using the fractional parts.
e.g.,

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \square$

Say, "How much is 5 groups of $\frac{1}{2}$?" (5 groups of $\frac{1}{2}$ equal $\frac{5}{2}$.) Ask a student to write $\frac{5}{2}$ as the answer to both examples.
e.g.,

$5 \times \frac{1}{2} = \frac{5}{2}$ $\qquad$ $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{5}{2}$

Say, "Look at the product, $\frac{5}{2}$. What kind of fraction is it? (improper) What is another name for $\frac{5}{2}$? Explain." ($\frac{5}{2}$ can be renamed as $2\frac{1}{2}$ because we can form 2 wholes and have $\frac{1}{2}$ left over when we put 5 halves together.) Ask a student to extend the multiplication expression as follows: $5 \times \frac{1}{2} = \frac{5}{2} = 2\frac{1}{2}$.

Repeat this procedure to multiply other whole numbers and fractions.

5.06.03

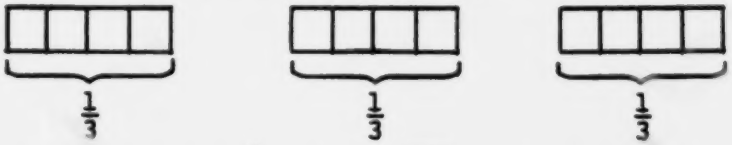
5.06.03.02 **find a fractional part of a whole number by creating equal parts (given a fraction whose denominator is a divisor of the whole number)**

STRATEGY 1:

Materials:
- interlocking cubes

Ask a student to interlock and display 12 cubes. Say, "We will use these cubes to find $\frac{1}{3}$ of 12. How do we find $\frac{1}{3}$ of anything?" (We divide it into 3 equal parts. Each part is called $\frac{1}{3}$.) Ask a student to divide the 12 cubes into 3 equal parts.

e.g.,



On the chalkboard write $\frac{1}{3}$ of 12 = $\square$. Ask a student to display $\frac{1}{3}$ of the cubes. Say, "How many cubes are in this stack that is $\frac{1}{3}$ of 12? (4 cubes) Do each of the other stacks have the same amount? Explain." (Each of the 3 stacks are equal. Each stack has 4 cubes.) Ask a student to complete the open sentence on the chalkboard, e.g., $\frac{1}{3}$ of 12 = $\boxed{4}$.

Say, "How can we find $\frac{2}{3}$ of 12? (We can count the number of cubes in 2 of the 3 groups.) How much is $\frac{2}{3}$ of 12?" ($\frac{2}{3}$ of 12 equals 8.) Ask a student to write this on the chalkboard, e.g., $\frac{2}{3}$ of 12 = 8. Repeat this procedure with other examples.

Note: Alternative materials to demonstrate this objective include: fractional parts, number lines, and paper folding.

5.06.03.03 **explain that the product of any unit fraction (fraction having a numerator of one) and a whole number can be interpreted as finding that fractional part of the number; and find such products when the whole number is a multiple of the denominator of the fraction**

STRATEGY 1:

Materials:
- interlocking cubes

5.06.03

5.06.03.04 **multiply a unit fraction by a second fraction by extending the understanding of taking a fractional part**

STRATEGY 1:

Materials:
- fractional parts

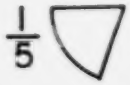
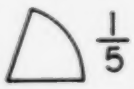
Say, "We have learned how to multiply a fraction and a whole number two ways. On the chalkboard write: $\frac{1}{2} \times 10 = \square$. Say, "What is one way we can find the product? (We can change the order of the factors to 10 times $\frac{1}{2}$. Then we can add 10 groups of $\frac{1}{2}$ to find the answer $\frac{10}{2}$ or 5.) What is another way to find the product? (We can find $\frac{1}{2}$ of 10 by separating 10 items into two equal groups and counting the number of items in 1 group. $\frac{1}{2}$ of 10 equals 5.) We will use this second method to help us understand how to multiply a pair of fractions where one fraction is a unit fraction." On the chalkboard write: $\frac{1}{2} \times \frac{10}{17} = \square$.

Say, "How much is $\frac{1}{2}$ of 10 cars? (5 cars) How much is $\frac{1}{2}$ of 10 people? (5 people) How much is $\frac{1}{2}$ of 10 thousand? (5 thousand) How much is $\frac{1}{2}$ of 10, regardless of the label, or the things we are talking about? (5) Look at the example on the chalkboard. How much is $\frac{1}{2}$ of 10 seventeenths?" (5 seventeenths) Complete the expression on the chalkboard, e.g., $\frac{1}{2} \times \frac{10}{17} = \frac{5}{17}$. Say, "To find $\frac{1}{2}$ of $\frac{10}{17}$, we can think of $\frac{1}{2}$ of 10, which is 5, and keep the denominator, 17. Let's see if this works with other fractions."

On the chalkboard write: $\frac{1}{3} \times \frac{3}{5} = \square$. Ask a student to display 3 fifths.

e.g., $\frac{3}{5}$

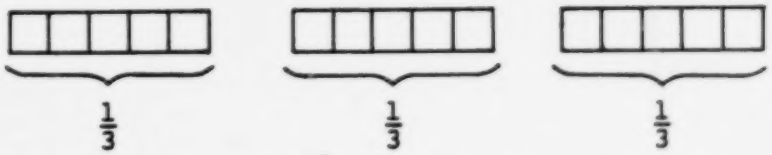
Say, "How can we use these fractional parts to find $\frac{1}{3}$ of $\frac{3}{5}$?" (We can separate them into 3 equal parts and take 1 of them.) Ask a student to demonstrate.

e.g.,  

STRATEGY 1: (cont'd)

Say, "We have learned how to find a fractional part of a whole number by creating equal parts." Display 15 connected cubes. Say, "How can we find $\frac{1}{3}$ of these 15 cubes?" (We can divide the 15 cubes into 3 equal groups. Each group will represent $\frac{1}{3}$ of the whole.) Ask a student to find $\frac{1}{3}$ of the 15 cubes.

e.g.,



Say, "How many cubes equal $\frac{1}{3}$ of 15?" (5 cubes) Write on the chalkboard and say, "$\frac{1}{3}$ of 15 = 5."

Say, "We can solve this another way." Below the equation on the chalkboard write: $\frac{1}{3} \times 15 = \square$.

Say, "When multiplying, does the order of the factors affect the product? Explain." (No, multiplication is commutative.) Ask a student to write this multiplication sentence on the chalkboard and change the order of the factors.

e.g.,

$$15 \times \frac{1}{3} = \square$$

Say, "How much is 15 times $\frac{1}{3}$? ($\frac{15}{3}$) Can we leave the answer in this form? Explain. (No, we change improper fractions to mixed numerals or whole numbers. $\frac{15}{3}$ equals 5.) If $15 \times \frac{1}{3} = 5$, what is the product of $\frac{1}{3} \times 15$? (5) What do you notice about the products?" (The products are the same. They are also the same as the number that is $\frac{1}{3}$ of 15.)

Repeat this procedure to relate multiplication to $\frac{1}{2}$ <u>of</u> 10 cubes, $\frac{1}{4}$ <u>of</u> 12 cubes, $\frac{1}{5}$ <u>of</u> 20 cubes...

e.g.,

$\frac{1}{2}$ of 10 = [5] $\quad$ $\frac{1}{4}$ of 12 = [3] $\quad$ $\frac{1}{5}$ of 20 = [4]

$\frac{1}{2} \times 10 =$ [5] $\quad$ $\frac{1}{4} \times 12 =$ [3] $\quad$ $\frac{1}{5} \times 20 =$ [4]

Elicit a generalization by saying, "How is each pair of examples alike? (Each pair has the same answer.) Why?" (Finding a product of a unit fraction and a whole number is the same as finding that fractional part of the whole number.)

STRATEGY 1: (cont'd)

Say, "How much is $\frac{1}{3}$ of $\frac{3}{5}$?" ($\frac{1}{5}$) Complete the expression at the chalkboard, e.g., $\frac{1}{3} \times \frac{3}{5} = \boxed{\frac{1}{5}}$. Say, "Now that we know the product, let's see if our rule works here. What did we say about finding $\frac{1}{2} \times \frac{10}{17}$? (We said we could simply find $\frac{1}{2}$ of the numerator 10, and keep the denominator, 17. Does that process work with $\frac{1}{3} \times \frac{3}{5}$? Explain." (Yes, $\frac{1}{3}$ of the numerator 3, is 1. If we keep the denominator 5, our answer is $\frac{1}{5}$. This is the same answer as we got when using the fractional parts.)

Repeat this procedure to multiply other pairs of fractions using the generalization that we can take the fractional part of the numerator and keep the denominator of the other factor.

On the chalkboard write: $\frac{1}{2} \times \frac{5}{9} = \square$. Say, "How is this example the same as the others we have solved? (We are asked to multiply a unit fraction times another fraction.) How is it different? (Finding $\frac{1}{2}$ of 5 does not give us a whole number to use as the numerator in the product.) Solving this example will be easier if we rename $\frac{5}{9}$ as an equivalent fraction with a numerator for which we can find $\frac{1}{2}$." Ask a student to rename $\frac{5}{9}$ on the chalkboard, e.g., $\frac{5}{9} \ \frac{x2}{x2} = \frac{10}{18}$. Say, "Is $\frac{10}{18}$ a useful equivalent for solving our example? Explain." (Yes, it is useful because we can easily find $\frac{1}{2}$ of 10, the new numerator.) Ask a student to complete the example on the chalkboard.

e.g.,

$$\frac{1}{2} \times \frac{5}{9} = \frac{1}{2} \times \frac{10}{18} = \frac{5}{18}$$

Say, "If $\frac{1}{2}$ times $\frac{10}{18}$ equals $\frac{5}{18}$, how much is $\frac{1}{2}$ times $\frac{5}{9}$? Explain." ($\frac{1}{2}$ times $\frac{5}{9}$ is also $\frac{5}{18}$ because $\frac{5}{9}$ and $\frac{10}{18}$ are equivalent.)

Repeat this procedure with other similar examples.

5.06.03.05 multiply <u>any</u> two fractions less than one by extending the understanding of taking a fractional part

STRATEGY 1:

Say, "We have learned how to multiply a unit fraction by a second fraction by taking a fractional part. Today we will learn how to

STRATEGY 1: (cont'd)

multiply any two fractions, unit and/or non-unit." On the chalkboard write: $\frac{2}{3} \times \frac{5}{8} = \square$. Say, "What is another way to find the product? (We can find $\frac{2}{3}$ of $\frac{5}{8}$.) Yes, an easy way to do this is to find $\frac{1}{3}$ of $\frac{5}{8}$. We can then double the product to find $\frac{2}{3}$ of $\frac{5}{8}$." On the chalkboard write:

$\frac{2}{3} \times \frac{5}{8} = 2 \times \frac{1}{3}$ of $\frac{5}{5}$.

Say, "We must now find $\frac{1}{3}$ of $\frac{5}{8}$. Since $\frac{1}{3}$ of the numerator 5 does not give us a whole number to use as a numerator, what must we do?" (We must rename $\frac{5}{8}$ as an equivalent fraction with a numerator for which we can find $\frac{1}{3}$.) Elicit equivalent fractions from the students, e.g., $\frac{10}{16}$, $\frac{15}{24}$, $\frac{20}{32}$... Say, "Which equivalent fraction has a numerator which will give us a whole number when we find $\frac{1}{3}$ of it? Explain. ($\frac{1}{3}$ of $\frac{15}{24}$ equals $\frac{5}{24}$.) If $\frac{1}{3}$ of $\frac{15}{24}$ equals $\frac{5}{24}$ then what is $\frac{2}{3}$ of $\frac{15}{24}$?" ($\frac{2}{3}$ of $\frac{15}{24}$ equals $\frac{10}{24}$.) Write the following on the chalkboard:

$$\frac{2}{3} \times \frac{5}{8} = 2 \times \frac{1}{3} \text{ of } \frac{5}{8}$$
$$= 2 \times \frac{1}{3} \text{ of } \frac{15}{24}$$
$$= 2 \times \frac{5}{24} = \frac{10}{24}$$

Say, "Is $\frac{10}{24}$ reduced to lowest terms? Explain. ($\frac{10}{24}$ is not reduced to lowest terms. The numerator 10 and the denominator 24 can both be divided by 2. $\frac{10}{24}$ reduced to the lowest terms is $\frac{5}{12}$.) The final line of the above equation should read: $= 2 \times \frac{5}{24} = \frac{10}{24}$ or $\frac{5}{12}$." Write this on the chalkboard. Ask the students to write the complete development of the equation in their notebooks.

Repeat this procedure to multiply any two fractions less than one.

5.06.03.06 make a picture or other graphic illustration to demonstrate multiplication of fractions less than one

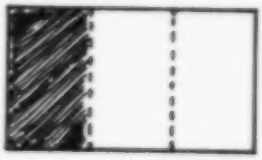
STRATEGY 1:

Materials:
- several sheets of unlined paper for each student or group of students
- crayons

STRATEGY 1: (cont'd)

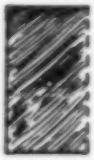
Distribute a sheet of paper and 2 different color crayons to each student or group of students. Say, "We will use these papers to help us further understand multiplication of fractions." On the chalkboard write: $\frac{1}{2} \times \frac{10}{3} = \square$. Ask a student to read this expression aloud. Say, "What is another way to think about this example? (We can think of $\frac{1}{2} \times \frac{1}{3}$ as $\frac{1}{2}$ <u>of</u> $\frac{1}{3}$.) Since we must find $\frac{1}{2}$ of $\frac{1}{3}$ let's fold our paper into thirds and shade $\frac{1}{3}$ of the paper so we can identify $\frac{1}{3}$." Demonstrate.

e.g.,



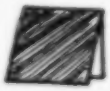
Say, "What does the problem ask us to find? (It asks us to find $\frac{1}{2}$ of $\frac{1}{3}$.) Refold your paper so that only the shaded third is showing." Demonstrate.

e.g.,



Say, "How can we find $\frac{1}{2}$ of this third? (We can fold this third in half.) Fold your paper in half." Demonstrate.

e.g.,



Say, "Using a different color, shade over the $\frac{1}{2}$ of $\frac{1}{3}$."

e.g.

Say, "What does this section, that is shaded in two colors, represent? (It represents $\frac{1}{2}$ of $\frac{1}{3}$.) Unfold your papers. Into how many equal parts is your paper now divided? (6 equal parts) Look at the part that is double shaded. What fractional part of the whole paper is it? (The double shaded part is $\frac{1}{6}$ of the whole paper.) How much is $\frac{1}{2}$ of $\frac{1}{3}$?" ($\frac{1}{2}$ of $\frac{1}{3}$ equals $\frac{1}{6}$.) On the chalkboard write: $\frac{1}{2} \times \frac{1}{3} = \frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$. Repeat this procedure to multiply other fractions.

STRATEGY 1: (cont'd)

e.g.,

$$\frac{2}{3} \times \frac{1}{4} = \frac{2}{3} \text{ of } \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$$

STRATEGY 2:

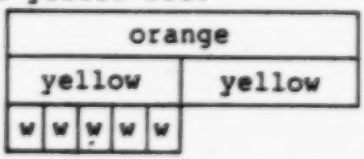
Materials:
- colored rods

Distribute a set of rods to each group of students. Say, "We will use these rods to demonstrate multiplication of fractions." On the chalkboard write: $\frac{2}{5} \times \frac{1}{2} = \square$. Say, "What does this example ask us to find?" (We must find $\frac{2}{5}$ of $\frac{1}{2}$.) Display an orange rod. Say, "If this rod represents 1 whole, which rod represents $\frac{1}{2}$ of the whole?" (a yellow rod) On the chalkboard write the following:

orange = 1 whole
yellow = $\frac{1}{2}$ of the whole

Say, "Now that we know what $\frac{1}{2}$ looks like, how can we find $\frac{2}{5}$ of $\frac{1}{2}$? (We must first find a rod that is $\frac{1}{5}$ of the yellow rod. Then we can simply use 2 of those to represent $\frac{2}{5}$ of $\frac{1}{2}$.) How can we find a rod that is $\frac{1}{5}$ of the yellow rod?" (We must find 5 rods of one color that equals the yellow rod. One of these will be $\frac{1}{5}$ of the yellow rod, two of them will be $\frac{2}{5}$ of the yellow rod.) Ask the students to find the 5 rods that equal the yellow rod.

e.g.,



Say, "What color rod is $\frac{1}{5}$ of yellow? (white) What is $\frac{2}{5}$ of the yellow rod?" (2 white rods equal $\frac{2}{5}$ of the yellow rod.) Add to the list of colors and values on the chalkboard.

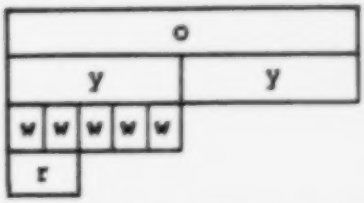
e.g.,

orange = 1 whole
yellow = $\frac{1}{2}$ of the whole
white = $\frac{1}{5}$ of the yellow rod

STRATEGY 2: (cont'd)

Say, "Since 2 white rods equal $\frac{2}{5}$ of the yellow rod, we can say that the amount represented by the 2 white rods is $\frac{2}{5}$ of $\frac{1}{2}$. Now we need to know this amount in terms of the whole. Find one rod equal to the 2 white rods. What is it?" (A red rod equals 2 white rods.)

e.g.,



Say, "How many red rods equal 1 whole orange rod? (5 red rods equal 1 whole orange rod.) Then what can we call the red rod? ($\frac{1}{5}$) Therefore, we can say that $\frac{2}{5}$ of $\frac{1}{2}$ equals $\frac{1}{5}$." On the chalkboard write:
$\frac{2}{5} \times \frac{1}{2} = \frac{2}{5}$ of $\frac{1}{2} = \frac{1}{5}$.

Repeat this procedure using other examples.

e.g.,

To find $\frac{1}{3} \times \frac{1}{3} = \square$

blue = I whole

green = $\frac{1}{3}$ of the whole

white = $\frac{1}{3}$ of the green

white = $\frac{1}{9}$ of the <u>whole</u> blue rod

Therefore, $\frac{1}{3} \times \frac{1}{3} = \frac{1}{3}$ of $\frac{1}{3} = \frac{1}{9}$.

5.06.03.07 multiply with fractions having values less than 1 by taking products of numerators and denominators

STRATEGY 1:

- sheets of blank paper for each student or group of students
- crayons

On the chalkboard write the following:

$\frac{1}{2} \times \frac{1}{2} = \square$ $\frac{1}{2} \times \frac{4}{5} = \square$ $\frac{1}{3} \times \frac{1}{4} = \square$ $\frac{2}{3} \times \frac{6}{8} = \square$

STRATEGY 1: (cont'd)

Say, "Can any of these examples be solved without paper folding or using another graphic illustration?" (Yes, the second and fourth examples can.) Ask a student to read the second example aloud. Say, "What are we asked to find? (We are asked to find $\frac{1}{2}$ of $\frac{4}{5}$. How much is $\frac{1}{2}$ of $\frac{4}{5}$? Explain." ($\frac{1}{2}$ of 4 is 2. Therefore, $\frac{1}{2}$ of $\frac{4}{5} = \frac{2}{5}$.) Ask a student to write this product on the chalkboard to complete the example.

Say, "Can any of the remaining examples on the chalkboard be solved in a similar manner? Explain. ($\frac{2}{3} \times \frac{6}{7}$ can be solved in a similar manner because we can easily find $\frac{2}{3}$ of 6.) How much is $\frac{2}{3}$ of $\frac{6}{7}$?" (Since $\frac{2}{3}$ of 6 is 4, then $\frac{2}{3}$ of $\frac{6}{7}$ is $\frac{4}{7}$.) Ask a student to write this product on the chalkboard to complete the example.

Say, "We can use the paper and crayons to solve the remaining two examples." To find these products, ask the students to use the paper folding method learned in P.O. 5.06.03.05, Strategy 1. Ask a student to write the products on the chalkboard to complete the examples.
e.g.,

$$\frac{1}{2} \times \frac{1}{2} = \boxed{\frac{1}{4}} \qquad \frac{1}{3} \times \frac{1}{4} = \boxed{\frac{1}{12}}$$

Say, "We will learn another way to solve examples with multiplication of fractions. Look at the first example. Since this is a multiplication example, can you find a way to use multiplication to solve it so that the answer is $\frac{1}{4}$? (If we multiply the numerators we get 1, which is the numerator of the product. If we multiply the denominators, we get 4, which is the denominator of the product.) Let's see if this works with the other examples. Look at the second example."
e.g.,

$$\frac{1}{2} \times \frac{4}{5} = \frac{2}{5}$$

Say, "Using the multiplication method you just described, what product do we get? (We got $\frac{4}{10}$ as the product.) Look at $\frac{4}{10}$. How can this answer be reduced to lower terms?" (If we divide the numerator and the denominator by 2, we will get the equivalent fraction, $\frac{2}{5}$.) Ask a student to show this on the chalkboard.
e.g.,

$$\frac{1}{2} \times \frac{4}{5} = \frac{4}{10} = \frac{2}{5}$$

STRATEGY 1: (cont'd)

Say, "Did the multiplication method work for this example? Explain." (Yes, however we had to reduce the product to lowest terms.)

Repeat this procedure with the remaining examples. Say, "What new way have we learned to multiply fractions?" (We can multiply the numerators and multiply the denominators. Then, if we can, we must reduce the product to lowest terms.)

5.06.03.08 create and solve word problems related to skills learned in this module

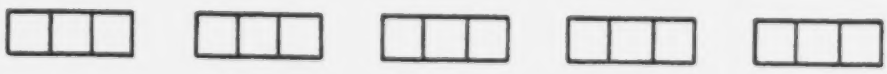
STRATEGY 1:

Materials:
- interlocking cubes

Display the following problem:

> There are 30 students in Jack's class. $\frac{1}{2}$ of them displayed their science projects at the fair. $\frac{3}{5}$ of the students who displayed their projects were girls. How many girls displayed projects?

Ask a student to connect 30 interlocking cubes to represent the 30 students. Say, "How can we use these cubes to help us solve the problem?" (We can separate the cubes into equal groups to find $\frac{1}{2}$ of 30.) Ask a student to separate the cubes to show $\frac{1}{2}$ of 30: Say, "What is $\frac{1}{2}$ of 30? ($\frac{1}{2}$ of 30 equals 15.) What does this represent? (This represents the number of students who participated in the science fair.) How can we use these 15 cubes to solve the problem?" (Since we know that 15 students participated, we must now find $\frac{3}{5}$ of 15. We can separate the 15 cubes into 5 equal parts. Each part will represent $\frac{1}{5}$.) Ask a student to demonstrate.
e.g.,



$\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$

STRATEGY 1: (cont'd)

Say, "How many cubes represent $\frac{1}{5}$ of 15? (3 cubes) How many cubes represent $\frac{3}{5}$ of 15? (9 cubes) What is the solution to the problem? Explain." (Since $\frac{3}{5}$ of 15 equals 9, we can say that 9 girls displayed their projects.)

Say, "There is another way to solve this problem. First we can find $\frac{3}{5}$ of $\frac{1}{2}$." On the chalkboard write: $\frac{3}{5}$ of $\frac{1}{2}$. Say, "What will this tell us?" (This will tell us the fractional name for the part of the group of students who are girls who displayed science projects at the fair.) Ask a student to find $\frac{3}{5}$ of $\frac{1}{2}$ as the other students write this in their notebooks, e.g., $\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$.

Say, "What must we now do to solve the problem? (We must find $\frac{3}{10}$ of 30 because $\frac{3}{10}$ of 30 represents the number of girls who displayed their projects.) How much is $\frac{1}{10}$ of 30? ($\frac{1}{10}$ of 30 equals 3.) Since $\frac{1}{10}$ of 30 = 3, how much is $\frac{3}{10}$ of 30? ($\frac{3}{10}$ of 30 equals 9.) Have we solved the problem? Explain." (Yes, 9 girls displayed projects.)

Repeat this procedure for other word problems, including some problems created by the students.

P.O. 5.06.03.01-.02

NAME: _____ CLASS: _____ DATE: _____

A. Write and solve a multiplication example for each group of fractions below. Reduce the answers to lowest terms.

① $\frac{2}{5} + \frac{2}{5}$ $2 \times \frac{2}{5} = \frac{4}{5}$

② $\frac{1}{12} + \frac{1}{12} + \frac{1}{12}$

③ $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

④ $\frac{3}{8} + \frac{3}{8} + \frac{3}{8}$

⑤ $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

⑥ $\frac{2}{9} + \frac{2}{9} + \frac{2}{9} + \frac{2}{9}$

B. Complete the examples below.

① $3 \times \frac{1}{8} =$ $\frac{3}{8}$

② $5 \times \frac{2}{3} =$

③ $2 \times \frac{1}{2} =$

④ $7 \times \frac{1}{4} =$

⑤ $10 \times \frac{3}{5} =$

⑥ $8 \times \frac{2}{5} =$

⑦ $5 \times \frac{1}{4} =$

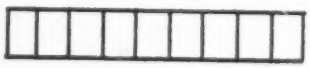
⑧ $2 \times \frac{3}{5} =$

⑨ $3 \times \frac{2}{7} =$

C. Use the pictures to solve each example.

①

$\frac{1}{2}$ of 8 = ☐

② 

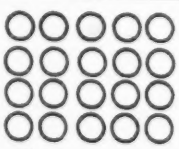
$\frac{2}{3}$ of 9 = ☐

③

$\frac{4}{5}$ of 15 = ☐

④

$\frac{1}{2}$ of 4 = ☐

⑤ 

$\frac{2}{5}$ of 20 = ☐

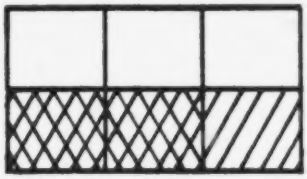
⑥

$\frac{3}{8}$ of 16 = ☐

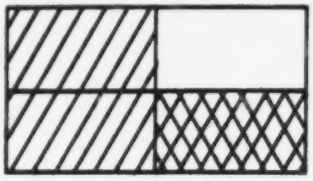
P.O. 5.06.03.06-.08

NAME: _____ CLASS: _____ DATE: _____

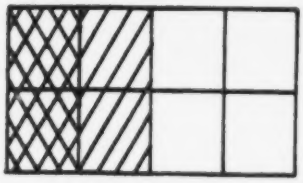
Write and solve the number sentence represented by each picture.

① 

$\frac{2}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$

② 

___ x ___ = ___

③ 

___ x ___ = ___

Multiply. Reduce your answers to lowest terms.

③ $\frac{1}{5} \times \frac{1}{2} =$ ☐

⑤ $\frac{3}{4} \times \frac{1}{8} =$ ☐

⑥ $\frac{1}{3} \times \frac{1}{5} =$ ☐

⑦ $\frac{2}{3} \times \frac{3}{5} =$ ☐

⑧ $\frac{5}{8} \times \frac{1}{2} =$ ☐

⑨ $\frac{4}{5} \times \frac{1}{3} =$ ☐

⑩ $\frac{5}{6} \times \frac{2}{3} =$ ☐

⑪ $\frac{3}{8} \times \frac{2}{3} =$ ☐

⑫ $\frac{1}{4} \times \frac{2}{9} =$ ☐

Solve the problems.

⑬ Mrs. Gold drives 60 miles to work each day. She stops for breakfast after driving $\frac{1}{3}$ of the way. How many miles does she drive before breakfast? _____

⑭ Mrs. Gold supervises 80 construction workers. $\frac{1}{2}$ of them are women. $\frac{1}{4}$ of the women were hired this year. How many women were hired this year? _____

Do you remember?

Round these numbers to the nearest 10,000.

45,721 rounds to _____

73,826 rounds to _____

52,063 rounds to _____

748,219 rounds to _____

688,385 rounds to _____

457,201 rounds to _____

P.O. 5.06.03.04-.05

NAME: _____ CLASS: _____ DATE: _____

A. Complete the multiplication examples below. Reduce the products to lowest terms.

① $\frac{1}{2} \times 10 = \boxed{5}$ So, $\frac{1}{2} \times \frac{10}{11} = \boxed{\frac{5}{11}}$	② $\frac{1}{3} \times 9 = \square$ So, $\frac{1}{3} \times \frac{9}{17} = \square$
③ $\frac{1}{5} \times 10 = \square$ So, $\frac{1}{5} \times \frac{10}{19} = \square$	④ $\frac{1}{4} \times 8 = \square$ So, $\frac{1}{4} \times \frac{8}{12} = \square$
⑤ $\frac{1}{10} \times 20 = \square$ So, $\frac{1}{10} \times \frac{20}{17} = \square$	⑥ $\frac{1}{6} \times 30 = \square$ So, $\frac{1}{6} \times \frac{30}{60} = \square$
⑦ $\frac{1}{5} \times 25 = \square$ So, $\frac{1}{5} \times \frac{25}{29} = \square$	⑧ $\frac{1}{2} \times 12 = \square$ So, $\frac{1}{2} \times \frac{12}{5} = \square$
⑨ $\frac{1}{3} \times 18 = \square$ So, $\frac{1}{3} \times \frac{18}{21} = \square$	⑩ $\frac{1}{8} \times 16 = \square$ So, $\frac{1}{8} \times \frac{16}{5} = \square$

B. Multiply. Reduce the products to lowest terms.

① $\frac{2}{3} \times \frac{3}{5} =$ $2 \times \frac{1}{3}$ of $\frac{3}{5} = 2 \times \frac{1}{5} = \frac{2}{5}$

② $\frac{3}{5} \times \frac{10}{11} =$ _____ = _____

③ $\frac{2}{9} \times \frac{18}{20} =$ _____ = _____

④ $\frac{5}{7} \times \frac{7}{10} =$ _____ = _____

⑤ $\frac{3}{4} \times \frac{12}{17} =$ _____ = _____

SPIRALING HOMEWORK

HOMEWORK

Questions to consider:

a. How long?

b. Should it be cyclical?

c. Should it be graded in difficulty?

d. Should it be adapted for the level of the class?

e. When should it be prepared?

f. How is it given to the student?

g. What if there no text/poor texts?

h. Is it reviewed in class?

i. Are all problems reviewed?

j. How much time is spent in class reviewing the homework?

k. Who writes the problems on the board?

l. Who corrects the wrong solution?

m. Is it collected?

n. Is it returned?

o. What part of the grade does homework reflect?

p. Is parent notified of missing homework?

q. What about students who never do the homework?

HOMEWORK

Research Findings:

More effective teachers assign homework and check homework regularly. Less effective teaching results if answers only are given to the previous day's assignment. Students should spend more than half of their time on tasks they can carry out with high success.

Questions to Consider:

Do I construct homework assignments in such a way as to provide each student with high success activities by including review questions?

Do I plan homework in a cyclical fashion so that students frequently review past topics?

Do I plan the homework assignments so that it includes both verbal and non-verbal problems?

Do I lightly stress the topic taught on a given day in the homework so that the teacher may determine the effectiveness of that day's lesson and continue to include this topic cyclically in future homework assignments?

Do I have all homework problems written on the board by students as other students are working on "Do Now" exercises?

Do I use the overhead projector for review of homework by having students prepare their solutions on transparencies?

Do the homework problems written on the board include the actual question as well as all steps in the problem's solution?

Do the students come to the board to explain their solutions and reasoning used in the problem, as well as to respond to students' questions?

Are the students in the class asked to compare their solutions with those at the board, asked to find the points of error, asked to explain the necessary changes to the student who is at the board doing the explanaton of his/her solution?

Do I view the student's organization of solutions to emphasize logical thought and to help other students reduce the amount of careless mistakes?

Are alternative solutions to problems investigated?

While reviewing homework, do I ask summary questions in order to assist students in comparing and contrasting procedures?

Do I create an atmosphere so that students are not fearful in making mistakes or not being able to do the complete solution?

Is the checking of homework an integral part of my instructional program?

ON HOMEWORK

HYPOTHESIS: Homework which reaches optimum significance needs to be carefully planned to harmonize with instructional goals. It should accent quality more than quantity. It should prevent wasted or uneconomical effort by students. It should be cyclical in nature to assure retention of previously learned material.

CHARACTERISTICS OF A POOR ASSIGNMENT:

1. The assignment consists of an arbitrary designation of pages or exercises in the text, chosen hurriedly and with little thought.

2. Prerequisite knowledge and skills have not been identified. There is no reason to believe the students can handle the material.

3. The assignment is fragmentary and bears little relation to what has gone before or what is to come after.

4. Although drill is not the goal, there is repetition to the point of frustration or boredom. No challenging exercises are present.

5. The assignment has not been planned for utilization in subsequent lessons; it doesn't set the stage for new ideas.

PLANNING VERSUS MAKING A HOMEWORK ASSIGNMENT:

Teachers should make a conscious effort in their planning:

a. to investigate future topics
b. to reinforce previous topics
c. to provide experiences that promotes discrimination among concepts in order to make the mathematics more available for the solution of the problems arising outside the mathematics classroom.

STEPS TO CONSIDER IN PREPARING CYCLICAL HOMEWORK ASSIGNMENTS:

1. The teacher should first list the topics that are to be taught each day (for the days between full-period tests).

2. The homework assignments are then planned <u>vertically</u> in the following manner: <u>ONLY TOPICS ARE LISTED FIRST.</u>
   a. the topic on the day's lesson is included for at least 3 successive days and then including it every other night or every third night, etc.
   b. The other topics chosen are from previous lessons alternating when they are given. Continuous computational topics should be included in most classes.

   c. After the topics have been listed for each assignment, the teacher should consider the length of the assignment and whether enough topics are covered for the needed variety on the test following these set of assignments. If adjustments are necessary, additional review from previous work can be included. Also movement of topics (because of too many long questions in one assignment versus not any long question, etc.) may be done at this point. Teachers are reminded that homework must be long enough to cover review but short enough to have students explain their solutions when it is gone over and yet allowing time for the teaching of a new developmental lesson each day.

d. After this planning, the books can now be used to select the problems. If this is done in a vertical fashion considering one topic and selecting problems for all of the assignments on this topic, gradation and variety on successive nights is made easier.

Each assignment should include at least 1 question that is verbal in nature.

Each assignment should contain in both single concept questions as well as multiple concept questions.

The new material should be lightly stressed in the homework. Giving 1 problem on the new topic is more successful than giving many. In this way, greater stress can be placed upon asking students to verbalize the rules, methods, or procedures needed to solve such a problem. If a student cannot do this problem, giving more than one problem will cause further frustration and math anxiety. If a student makes an error, he would reinforce this error many times if more than one problem is given. Giving one problem also provides a quick way of determining whether the new material was learned. Variations of the topic are given in successive nights.

e. Each assignment includes pages to be read on the topic taught that particular day.

f. Teachers should realize that if there is any uncertainty of completing a topic on a particular day, it is better not to include this topic on that night's assignment.

ADDITIONAL COMMENTS WITH REFERENCE TO HOMEWORK:

1. If homework assignments are planned from test to test,

   a. teachers will not only know whether they have enough topics for a varied and comprehensive test but will also know if they have both long and short topics for the given test.

   b. teachers can include review of topics previously tested that in their judgment need further review spirally in future homeworks and then include these topics on the next test. This should help the student who was unsuccessful on a given topic when it was first tested become confident when they are successful on the next test.

   c. students become more familiar with a comprehensive test and not as thrown when they take a state exam or a city-wide exam.

   d. the teacher can make up the test at the same time that they have prepared the set of homeworks between tests. It this way the test can be written, duplicated, proofread and a key prepared well ahead of it being given. Teachers are then directed as to what schedule of teaching should occur and in some way prevents going off on a tangent in class.

2. The overhead projector is very helpful in going over the homework. Students may purchase china markers at the beginning of the term. The teacher then gives an acetate when assigning problems to each student to bring in the following day. The student then comes to the machine, explains his/her work, chairing any questions asked from the class. This furthers verbalization by students and provides more sustained explanations. Teachers need to stress that students give explanations of what was one or thought of to go from step to step rather than have them read the work. No time is spent putting up the problems

on the board. All work becomes visable to the entire class. More student interaction takes place since this student is chairing questions from the class. Students may borrow the acetate and bring in the next day. If a disruption during a period (fire drill, etc.) the homework may be gone over the following day. Time may be spent on training students to pay attention to the discussion on hand not to copying of the work. More emphasis can be placed on re-doing work done incorrectly rather than mere copying of the solution. Going over the homework after the applications of the new lesson and before the summary of the lesson usually speeds up the going over of the homework rather than extending if to the entire period or most of the period.

3. Teachers are asked to prepare all solutions to each assignment in advance of their need with the students. In this way the teacher can determine the types of errors that could occur and know what to stress in the teaching of the topic. Time is saved since the teacher can quickly determine the point of error and ask questions accordingly.

4. Teachers may prepare homework sheets for distribution to the class. The benefits of having such a sheet are as follows:

   a. for students:
      (1) if absent, they have the homework and reading assignment and can come in prepared
      (2) provides information as to what topics to study for a test
      (3) gives enough advanced notice of test date to allow for planned study periods
      (4) if they know of another commitment for a particular night, they can do their homework in advance

   b. for teachers:
      (1) eliminates need for writing assignment on the board at the beginning of each class period
      (2) eliminates nightly pressure
      (3) eliminates having to first review which problems were already given
      (4) allows for gradation of problems
      (5) allows for a better planned test
      (6) provides material to be given to new admits and those students who return after extended absence.

   c. for parents:
      (1) gets better image of school, department and teacher since they see more planning
      (2) appreciates that child can do work if absent
      (3) appreciates not having last minute notice of test

Sample model of information on homework sheet:

#23 Given: Due:

Name of Text	104	201	204	212	277	313	Page #
Read 236-8,249, 250,305,306	46	15	48	13	19	16	Problem #

Answers to problems:

Note:

Putting answers on sheet helps student learn how to use answers when doing solutions and reduces math anxiety.

THE FOLLOWING MAY BE DISCUSSED WITH STUDENTS AND GIVEN TO THEM IN PRINTED FORM FOR REFERRAL AT HOME WHEN DOING THEIR MATHEMATICS HOMEWORK:

HOW TO DO YOUR MATHEMATICS HOMEWORK

PURPOSES OF HOMEWORK:

a. To help reinforce what you learned in class.
b. To help identify those parts of the lesson you do not understand.
c. To increase the number of correctly solved problems to which you can refer when review ing for a test.

1. Read the introductory section of the assignment to identify and understand the principle it teaches.

   a. Make certain you understand the meaning of every word in the section.
   b. Visualize every step along the way.

2. Do the sample exercises.
   This will determine how well you understand the principles being taught and whether you will be able to work problems based on those principles.

   a. Read each step carefully in the model problems.
   b. Understand every term used.
   c. Follow the reasoning behind each manipulation.
   d. Solve the problem all over again on your own.

   It is a good idea to reserve a section of your loose-leaf notebook for working through the sample problems in the book.
   Writing out the steps is far better than merely reading through what the author has written.
   This section in your notebook will be a handy review for you before tests.

3. Recite the principles you have learned. If you cannot repeat the principles used, go back to the reading material prior to the exercises and read it through again.
   Write the principles and procedures used to solve each type of problem on an index card from which you can study for a test as well as quickly review previous work each day.
   Make review a part of your daily calendar. It will eliminate last minute cramming and further retention of your mathematical skills.

What to do if you "get stuck" on a problem!

a. Look back at the book, at your notes, and at the sample model problems.

What to do when an answer is wrong in your homework.

a. Analyze the problem to find out at what point the error was made.
b. Listen to the analysis done in class.
c. Take down the correct answer.
d. At home, re-do the problems you got wrong before trying the new assignment.

ENCOURAGING STUDENTS TO ASK QUALITY QUESTIONS

Creating Thought Provoking Questions

Tetra food: 1 kilogram for $1.20
Goldfish food: 2 kilograms for $2.00

Gravel: 2 kilograms for $1.50
5 kilograms for $3.00

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

WHAT QUESTIONS CAN YOU CREATE FROM SAM'S SPORTS SALE CIRCULAR?

1. _____
2. _____
3. _____
4. _____
5. _____
6. _____

ACTIVE LEARNING

Research Findings:

Effective teachers try to keep most of their students from slipping into passive learning. The higher the number of student-initiated questions and comments, the higher was student achievement in mathematics.

Questions to Consider:

Do I ask questions that call for a more sustained student response?

Do I have students play an active part in the explanations to class problems, homework problems and test problems i.e., by having students come to the board to explain the process used and react to other student responses?

Do I return student questions to the other members of the class for reactions, corrections, additions, etc?

Do the students feel comfortable to ask questions?

Do I plan the "Do Now" exercise so that it is a review of the pre-requisite skills, definitions and procedures needed for the new lesson of the day.?

Do I plan a "Do Now" exercise so that it is attainable by most students and can be discussed shortly after the period begins?

Do I use the developmental approach in teaching by having students discover rules and procedures and make generalizatons?

MODELING

Research Findings:

There is a clear correlation between the frequency of mathematics teachers' use of conditional reasoning in their speech (e.g., "if-then" and "whether-then" statements), and their students understanding of logical statements. Students need ample opportunities to use logic and language.

The use of counter-examples of mathematical statements, in conjunction with (positive) example of those statements, resulted in higher achievement than did the use of examples alone.

Questions to Consider:

Do I emphasize to the students the need for knowing both what methods to use in solving a problem as well as when to use the particular method?

Do I have students either use the "if-then" form when responding or give complete sentence responses?

Do I require students to write justifications to their homework solutions and class problems in the "if-then" format?

Do I not interpret students' responses but provide a counter-example for student reaction?

Do I encourage student reactions to student responses?

Do I ask students to give counter-examples to prove or disprove their conjectures?

TIME ON TASK

Research Findings:

More effective teachers use about three fourths of each period for demonstration and discussion, and only about one-fourth of each period for individual seatwork.

Questions to Consider:

Do I prepare solutions for homework and class problems in advance so that a comparison of the students work may be made more quickly?

Do I have students write and explain reasons to justify their solutions to problems?

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REASONING SKILLS PROJECT
Mathematics and Communication Arts

ENCOURAGING STUDENTS TO ASK QUALITY QUESTIONS

The ability to pose excellent questions is of key value to success as a problem solver. As we aim to help our students become better thinkers and problem solvers, we must find a way to help them to ask quality questions.

The first step in this effort is to learn to pose good questions ourselves. This step is necessary but not sufficient.

Research has shown that students who regularly <u>hear</u> good questions do not themselves ask better questions as a result; when compared to other students who do not have the opportunity to hear good questions, they show no significantly better ability. We conclude that <u>modelling</u> good questioning behavior is not enough by itself to help our students master the skill of questioning. We must take special steps if we seriously intend to help our students grow in this area.

<u>We hope to elicit</u> questions such as these:

- What is the difference between ____ and ____?
- Could you provide an example which is ____ but not ____?
- Why is ____ better than ____?
- Which way is the best?
- How did you know you should ____?

Ways To Encourage Students To Ask Stimulating Questions

We believe that we can stimulate questions such as these if we do the following:

1. Encourage student-student conversation.

   a. Don't interrupt student-student exchanges by interjecting our own comments.

   b. Train students to comment on the prior remark before adding something of their own.

   c. Give the students the opportunity to present their own work (e.g., to explain their solutions to homework problems: an overhead projector can be used to increase the effectiveness and efficiency of this procedure).

   d. Encourage students to explain their ideas by asking:

      HOW DID YOU REACH THAT CONCLUSION?
      WHY DO YOU THINK SO?

2. Talk about questions; explain why particular questions have greater value than others.

3. Provide problems and situations about which interesting questions can be posed.

4. Provide sufficient time to formulate quality questions. While trying to elicit a quality question, do not call on the first student who raises his hand; this would rob the others of the valuable experience of formulating their own questions. Instead, while waiting for the others to reach their own insights, go over to the student who has an idea already and ask him to write it down so you can see it.

5. Encourage the students to make up new problems.
   Ask: What new ideas came up today?
   Assign: Make up a (non-computational) problem based on (this investigation).
   Award credit for original problems, as well as original solutions.

6. Encourage the students to reflect on impediments they encounter while attempting to solve problems.
   Assign: Annotate your homework with a discussion of the errors you made.
   Assign: Annotate this returned exam with a discussion of the errors you made.

**** 7. ASK: WHAT QUESTION SHOULD WE ASK NEXT?

DEVELOPING INSTRUCTIONAL

APPROACHES FOR TEACHING

DECIMALS

BINGO

0.3	0.4	0.7
0.8	0.1	0.6
0.5	0.9	0.2

0.3	0.6	0.4
0.9	0.5	0.2
0.1	0.8	0.7

0.9	0.7	0.5
0.4	0.2	0.1
0.8	0.3	0.6

0.6	0.7	0.4
0.1	0.9	0.8
0.2	0.5	0.3

BINGO

THREE TENTHS	FOUR TENTHS	SEVEN TENTHS
EIGHT TENTHS	ONE TENTH	SIX TENTHS
FIVE TENTHS	NINE TENTHS	TWO TENTHS

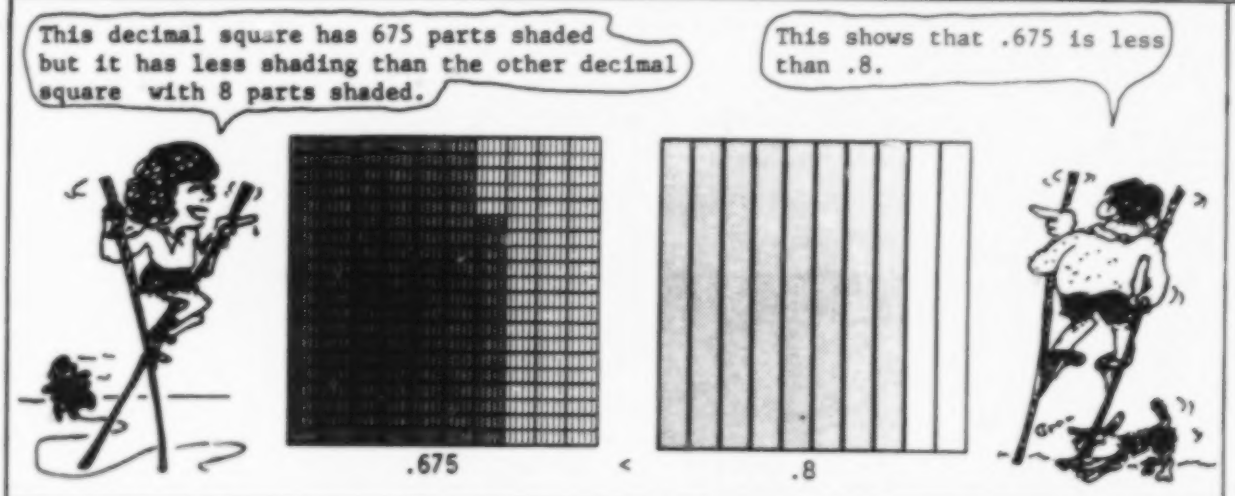
NINE TENTHS	SEVEN TENTHS	FIVE TENTHS
FOUR TENTHS	TWO TENTHS	ONE TENTH
EIGHT TENTHS	THREE TENTHS	SIX TENTHS

THREE TENTHS	SIX TENTHS	FOUR TENTHS
NINE TENTHS	FIVE TENTHS	TWO TENTHS
ONE TENTH	EIGHT TENTHS	SEVEN TENTHS

SIX TENTHS	SEVEN TENTHS	FOUR TENTHS
ONE TENTH	NINE TENTHS	EIGHT TENTHS
TWO TENTHS	FIVE TENTHS	THREE TENTHS

Contrasting Decimals

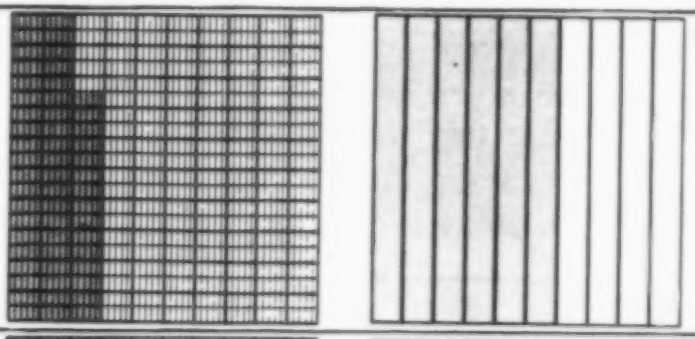
Name _____ Date _____


This decimal square has 675 parts shaded but it has less shading than the other decimal square with 8 parts shaded.
This shows that .675 is less than .8.
.675
<
.8

For each pair of squares, write decimals in the blanks to complete the two inequalities.

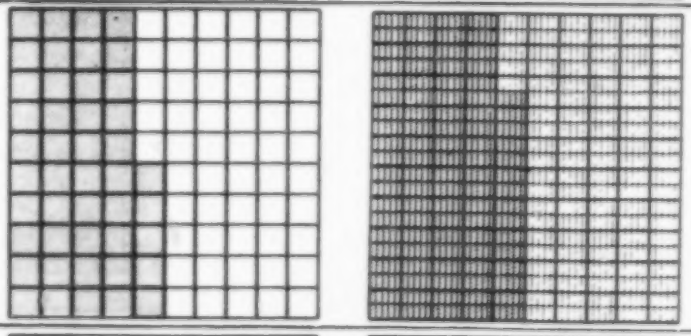
1. _____ > _____

2. _____ < _____



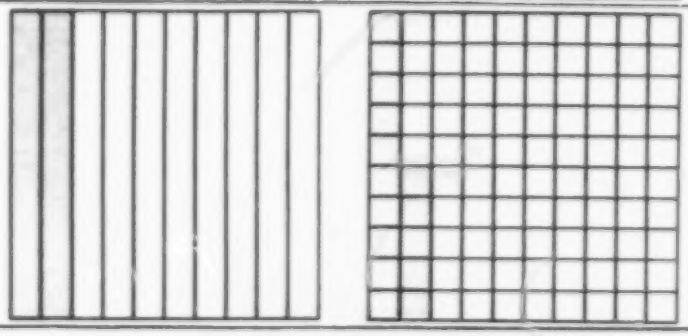
3. _____ > _____

4. _____ < _____



5. _____ > _____

6. _____ < _____



PROBLEM SOLVING

DECEMBER

1

In a twelve hour period of time, for how many minutes is the sum of the digits on an electronic digital clock greater than 15?

2 1792 b. **Nikolai Lobachevsky,** developed non-Euclidean geometry.

3

1616 b. **John Wallis,** produced the result

$$\frac{4}{\pi} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdots$$

4

2, 9, 28, 65, 126, 217, ___?

5

Mr. Pennypacker left half of his estate to his wife, $40 000 to his daughter, half of what remained to his butler, half of what remained for the care of his favorite dog, and the remaining $6000 to charity. What was the value of Pennypacker's estate?

6

415 of the 700 people interviewed said the they liked only chocolate nut ice cream; 269 said that they liked only blackberry; 123 liked both flavors. How many of the people interviewed didn't like either flavor?

7

What's the number?

It's a three digit perfect square and a perfect cube.

8

1865 b. **Jacques Hadamard,** leading mathematician of the 20th century, contributed to most branches of mathematics, especially complex analysis.

9

How old am I?

My age is a perfect square number of years greater than a perfect square; it is also a perfect square number of years less than a perfect square. Next year my age will be a prime number less than 50.

10

1804 b. **Karl Jacob Jacobi,** worked on the theory of determinants.

1816 b. **Ada Byron Lovelace,** colleague of Babbage, first wrote about computer programming.

11

How many perfect squares can be displayed on an 8 digit calculator?

12

1823 b. **Leopold Kronecker,** worked in the field of algebra.

13

How many numbers between 1/5 and 1/6 can be displayed on an 8 digit calculator?

14

1548 b. **Tycho Brahe,** indicated by his precise astronomical measurements that Aristotle's views of the planets were incorrect.

15

1802 b. **János Bolyai,** discovered principles of non-Euclidean geometry.

16

The Boston Tea Party took place on this date in the 18th century in the year ___?

*The sum of the digits of the year is 18.

*The ones digit is equal to $\sqrt[3]{243}$.

17

1842 b. **Marius Sophus Lie,** worked in differential equations, group theory.

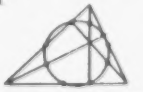
1706 b. **Emilie du Châtelet,** translated *Newton's Principia* into French.

18

Which two consecutive numbers have squares that differ by 75?

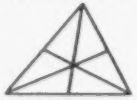
19

1783 b. **C. J. Brianchon,** first used "nine point circle." "Brianchon's theorem."



20

1648 b. **Tommaso Ceva,** interested in geometry, geometric harmonic means, poetry.



21

A five-digit number is chosen at random from a table of all five-digit numbers. What is the probability that the number is evenly divisible by 1, 2, 3, 4, and 5?

22

1887 b. **Srinivasa Ramanujan,** contributed to theory of numbers, continued fractions.

23

If a three digit number is chosen at random from a set of all three digit numbers, what is the probability that all three digits will be prime?

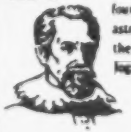
24

1822 b. **Charles Hermite,** proved the transcendence of *e*.

25 1642 b. **Sir Isaac Newton,** major contributor to mathematics, physics, astronomy, natural philosophy.

26 1792 b. **Charles Babbage,** worked on an elaborate calculating machine.

27 1571 b. **Johann Kepler,** a founder of modern astronomy, advanced the use of logarithms.



28 1903 b. **John von Neumann,** developed game theory and applications of economics.

29

1856 b. **Thomas Jan Stieltjes,** generalized Reimann integral in calculus.

30

What is the probability that when the letters in the word *mathematics* are arranged at random, they will spell *mathematics*?

31

1864 b. **Robert Grant Aitken,** mathematician, astronomer, observer of double stars.

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OFFICIAL NEWSLETTER OF NEW YORK STATE ASSOCIATION FOR SUPERVISION AND CURRICULUM DEVELOPMENT

JUNE 1987 VOL. 27, NO. 4

Teaching of Thinking Fails to Teach Thinking

Although current texts are now full of thinking skills, kids are still not being taught to think. So asserted Robert Sternberg of Yale University and author of *Intelligence Applied* in his speech "Teaching Thinking to Children — Where are the Gaps?" One problem is the way most textbooks present thinking skills — the relevant section is usually separated from the rest of the material, often at the end of the chapter, easy to leave out.

Most significant, says Sternberg, are the ways that thinking problems and questions on intelligence tests differ from everyday, real-life problems. Sternberg outlined the major differences.

- Most book problems require one isolated skill while real problems require several skills in combination. Thus, students learn individual skills but miss the essential "when to use a skill and how to combine it with others."
- In books, problems are all set up, yet in real life a major part of problem solving is recognizing that there is a problem and defining the problem. Students need, says Sternberg, to learn to recognize and define problems for themselves.
- Artificial problems usually have only one path to solution; on the other hand, everyday problems are not well-structured; most afford a variety of possible solutions.
- Schoolbook problems lack the context of real life. In life, solutions interact with life. A change in context can drastically alter the solution.
- Tests and book problems favor the use of formal knowledge and reasoning while everyday issues often demand informal knowledge and methods. Indeed says Sternberg, studies show that good formal reasoners are not necessarily good informal reasoners; the correlation is very low.
- Formal tests have answers that demand the seeking of evidence to confirm a pattern or belief system. Life problems, however, require the opposite. In order to move forward, people need to look for the evidence that does not confirm the hypothesis but rather opens up new hypotheses.
- Real problems, unlike school problems, have consequences, and people solve consequential and inconsequential problems in very different ways.

Robert Sternberg

- In life people work together in groups to solve problems; by contrast, in school, most collaborative effort is viewed as cheating.

Thus, insists Sternberg, if our intention is to educate children to be good thinkers in life, our texts and tests go about it in the wrong way. Everyday problems can be "complex, messy and persistent." Educators need to infuse more of real life into the teaching of thinking. Yet, it is extremely important to drive home the connection between the real problems and the thinking skills used to solve them, so the lessons do not deteriorate into mere discussions of problems. Sternberg concludes that educators need to keep working to improve the teaching of thinking, standardized tests and intelligence tests so that what we demand and what we measure are skills for life and not just skills for school.

262

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PROBLEM EXPLORATION STRATEGIES

1. State the goals of a problem clearly.
2. Maintain an overview.
3. Work backward.
4. Restate the problem to eliminate ambiguities and clarify objectives.
5. Look for hidden assumptions.
6. Adopt a different point of view.
7. Consider a related but simpler problem.
8. Consider special cases.
9. Consider extreme cases.
10. Use Analogy.
11. Find a real or an abstract model for a problem.
12. Carry out a simulation.
13. Compare and contrast.
14. Systematically account for all possibilities.
15. Consider the consequences of an idea.
16. Eliminate the impossible but not the improbable.
17. Look for a pattern.
18. Use inference.
19. Generalize conclusions carefully.
20. Recognize and use symmetry.
21. Make a diagram.
22. Keep good records.
23. Estimate.
24. Use trial-and-error.
25. Prioritize goals.
26. Sequence ideas.
27. Find an alternative solution.

HOW TO SOLVE IT

First.
You have to *understand* the problem.

UNDERSTANDING THE PROBLEM

- *What is the unknown? What are the data? What is the condition?*
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

Second.
Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

DEVISING A PLAN

- Have you seen it before? Or have you seen the same problem in a slightly different form?
- *Do you know a related problem?* Do you know a theorem that could be useful?
- *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Third.
Carry out your plan.

CARRYING OUT THE PLAN

- Carrying out your plan of the solution *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

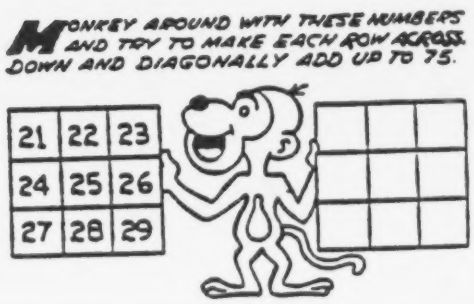
Fourth.
Examine the solution obtained.

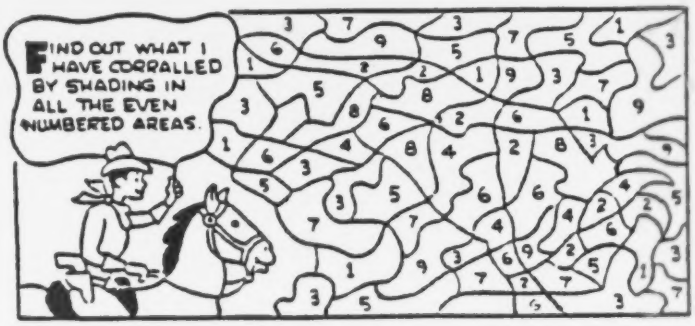
LOOKING BACK

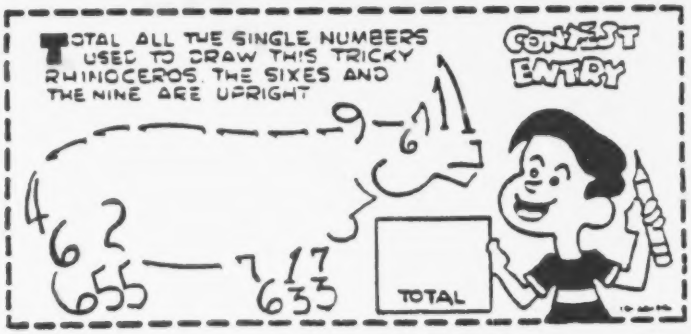
- Can you *check the result?* Can you check the argument?
- Can you derive the result differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

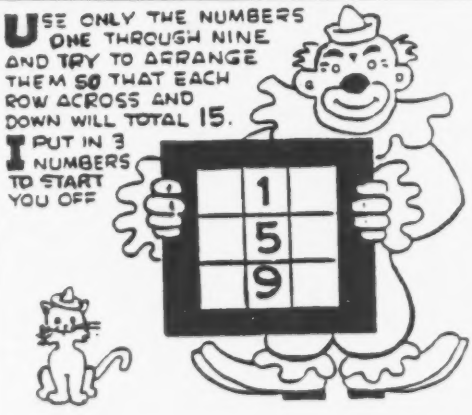
"PROBLEM OF THE DAY" ACTIVITIES

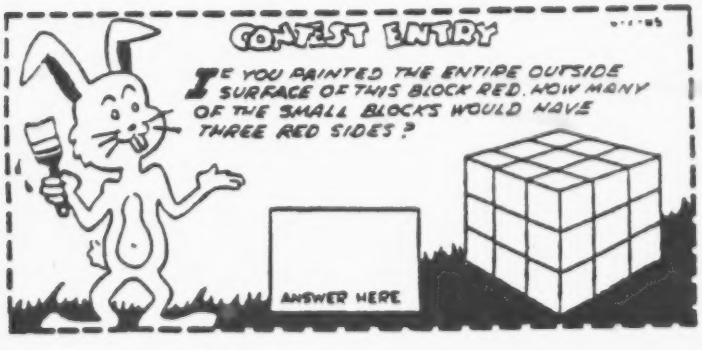
Samples from the N.Y. Post (Saturdays) and the Daily News (Sundays)

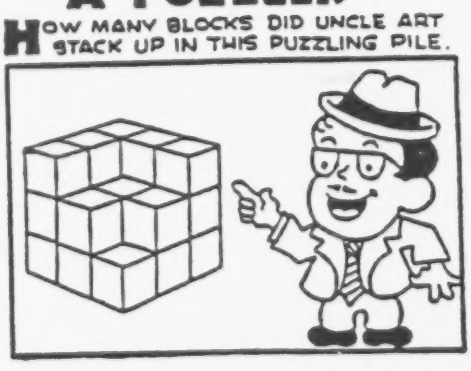







USE ONLY THE NUMBERS ONE THROUGH NINE AND TRY TO ARRANGE THEM SO THAT EACH ROW ACROSS AND DOWN WILL TOTAL 15.
I PUT IN 3 NUMBERS TO START YOU OFF
1
5
9


CONTEST ENTRY
IF YOU PAINTED THE ENTIRE OUTSIDE SURFACE OF THIS BLOCK RED, HOW MANY OF THE SMALL BLOCKS WOULD HAVE THREE RED SIDES?
ANSWER HERE



A Sample of a MAGIC SQUARE

179-C

Draw the magic square and complete it. The three numbers in each row and column should add to 15.

5.9		7.7	= 15
			= 15
2.3		4.1	= 15
= 15	= 15	= 15	

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INCORPORATING PROBLEM SOLVING SKILLS IN YOUR CLASSROOM

The more opportunities that students have for solving problems, the more skillful they will became as problem solvers. Time should be allotted during each class lesson to the skills of problem solving as well as to their application. Some suggestions are:

- Have students prepare their own original problems from a given set of data.
- Start each lesson with a problem solving experience.
- Be sure that each homework assignment contains at least one problem solving activity.
- Start a class book of problems. Bring in charts, menus, price lists, pictures, listings of information, graphs and tables gathered from newspapers and magazines. Ask pupils to write and solve problems related to these materials.
- Classroom activities such as collecting or spending money, team scores, cake sales, health data, planning for a trip or party opportunities for generating and solving verbal problems.
- Use student prepared problems for homework. Identify the problems with the student's name.
- Have students who wrote the problem check the answers submitted.

EFFECTIVE TEACHING OF PROBLEM SOLVING

When visiting a classroom in which the learning of problem solving is taking place, one would expect the teacher to:

1. have carefully planned the lesson.
2. encourage students to verbalize their thinking.
3. ask a sequence of questions which lead students to discover a concept.
4. foster an exchange of ideas among students.
5. create the impression that the teacher is also investigating the problem.
6. encourage students to explore original ways of solving problems.
7. make clear that a variety of ways exist to solve problems by asking, "Can we think of another way of doing this?"
8. focus students' attention on the process of thinking.
9. promote divergent thinking.
10. avoid lecturing.
11. avoid asking "Are there any questions?" or "Is there anyone who doesn't understand?"
12. avoid allowing students to shout out answers spontaneously.
13. avoid using responses of more able students to dictate pace of class.
14. ask class to react to an answer given by a student rather than teacher indicating approval or disapproval.
15. encourage students to defend or justify answers.
16. encourage students to estimate and determine reasonableness of answers.
17. show more interest in the process of solving a problem rather than the answer.
18. motivate students to want to think.
19. encourage students to ask questions.
20. give students time to think.
21. listen to students.
22. offer activities which permit students to explore skills and/or concepts before trying to find a solution.

STRATEGIES FOR IMPROVING THE CLASSROOM CLIMATE FOR PROBLEM SOLVING

1. Emphasize ideas such as
   - problem solving takes time and patience
   - it is OK to make mistakes; sometimes errors help one arrive at a solution
   - a student who makes progress toward a solution has been successful
   - it's really fun and satisfying when you try hard and solve a problem
2. Give a story or a problem and have students act out the situation and phrase the question in their own words. Use objects when helpful.
3. Give students a number sentence and encourage them to formulate a problem that can be solved using the sentence. Post student's problems on a bulletin board for other students to solve, or make a weekly newsletter containing problems that students have created for others to solve.
4. Ask questions to help students understand the problem better.
5. Ask questions that give the students hints for solving when needed.
6. Ask questions that encourage the students to describe and analyze their solutions.
7. Compliment students for taking time to think through a problem rather than for how quickly they find a solution.
8. Use detective or mystery stories as an aid to help students problem-solving skills.
9. Look for everyday situations about which you can pose questions that stimulate thinking. e.g. At about what time will the hands on a clock be at right angles to each other?
10. Have students explain their plan and solution.
11. Have students look for and discuss alternate solutions to the same problem.
12. Help students build problem-solving skills by making new problems from a given problem. e.g. Change one or more of the conditions of the problem; interchange the "given" and the "to find".
13. Personalize problems. Use the names of your students and of local places to make problems come alive.
14. Encourage students to choose things they are interested in and write problems about them for other students to solve.
15. Have students work in groups and solve problems so that they can discuss the strategies possible to solve the problems.
16. Do not reject guesses; pursue them.

MATHEMATICS
K - 8

PROBABILITY
FRACTIONS
DECIMALS
PROBLEM SOLVING

NEW YORK CITY BOARD OF EDUCATION
Office of Curriculum Development and Support
Division of Curriculum and Instruction

FRACTIONS

Topic 06

0.06.01 **A whole unit**
The student will be able to . . .
- identify an object as a unit when so considered
- identify a part of a whole
- distinguish part of a unit from the whole unit

0.06.02 **Share objects**
The student will be able to . . .
- recognize that when a group is separated into two parts of the same size, each part is 1/2 of the whole group
- solve oral story problems related to skills learned in this module

0.06.03 **Compare a half unit to a whole unit**
The student will be able to . . .
- recognize that a whole object is greater than one-half (1/2) of the object
- recognize that one-half (1/2) of an object is less than a whole

0.06.04 **The fraction one-half (1/2)**
The student will be able to . . .
- recognize equal parts of a whole
- recognize that when an object is separated into two (2) pieces of the same size, each piece is one-half (1/2) of the whole object
- divide representative objects into halves
- solve oral story problems related to skills learned in this module

1.06.01 **Halves**
The student will be able to . . .
- separate representative objects into halves (B)
- identify, write and record the symbol for one-half as 1/2, for two-halves as 2/2 (B)
- solve story problems related to skills learned in this module (B)

1.06.02 **Fourths**
The student will be able to . . .
- separate representative objects into fourths (B)
- identify and write fractional notation for one-fourth as 1/4, for two-fourths as 2/4, for three-fourths as 3/4, for four-fourths as 4/4 (B)
- identify 1/2 as equal to 2/4 pictorially, and record using the equal symbol (=) (B)
- solve story problems related to skills learned in this module (B)

1.06.03 **Thirds**
The student will be able to . . .
- separate representative objects into thirds (B)
- identify, write and record fractional notation for one-third as 1/3, for two-thirds as 2/3, for three-thirds as 3/3 (B)
- solve story problems related to skills learned in this module (B)

2.06.01 **Fractions — fifths, sixths, and tenths**
The student will be able to . . .
- recognize that when an object is separated into parts of equal size, each part may be written as a unit fraction of the whole object (A)
- separate a concrete object into fifths, sixths, and tenths and identify the parts as equal (A)
- identify and write fractional notation for fifths through 5/5, sixths through 6/6 and tenths through 10/10 (A)
- solve story problems related to skills learned in this module (A)

2.06.02 **Identify fractional parts of a group (1/2, 1/3, 1/4)**
The student will be able to . . .
- separate a group through 10 objects into two equal groups using fractional parts (A)
- separate a group through 20 objects into four equal groups using fractional parts (A)
- separate a group through 15 objects into three equal groups using fractional parts (A)
- solve story problems related to skills learned in this module (A)

3.06.01 **Equivalent fractions**
The student will be able to . . .
- identify and define the numerator and denominator in a given fraction (A)
- identify and record the total number of parts to equal one whole (halves, thirds, fourths, sixths and eighths) (A)
- compare and order like fractions (A)
- compare and order unlike unit fractions (A)

3.06.02 **Non-unit fractional parts of a group**
The student will be able to...
- separate a number of objects into non-unit fractional parts (B)
- solve word problems related to skills learned in this module (B)

3.06.03 **Addition and subtraction of fractions with like denominators**
The student will be able to . . .
- add fractions having like denominators in vertical and horizontal formats (B)
- subtract fractions having like denominators in horizontal and vertical formats (B)
- solve story problems related to skills learned in this module (B)

3.06.04 **Improper fractions and mixed numbers**
The student will be able to . . .
- define and identify an improper fraction (B)
- define and identify a mixed number (B)
- change an improper fraction to a mixed number or a whole number (B)
- change a mixed number or a whole number to an improper fraction (B)

3.06.05 **Ratio**
The student will be able to . . .
- write a ratio to represent a relationship between two quantities (B)
- solve word problems related to skills learned in this module (B)

4.06.01 **Addition and subtraction of fractions with common denominators**
The student will be able to . . .
- add fractions with common denominators (A)
- subtract fractions with common denominators (A)
- solve word problems related to skills learned in this module (A)

4.06.02 **Multiplication of fractions**
The student will be able to . . .
- multiply fractions having values less than 1 (B)
- rename any whole number as a fraction with a denominator of 1 (B)
- multiply a whole number by a fraction (B)

4.06.03 **Equivalent fractions**
The student will be able to . . .
- recognize equivalent fractions (A)
- match a set of equivalent fractions with its corresponding points on the number line (A)
- use the multiplication identity to generate equivalent fractions (B)

4.06.04 **Order of unit fractions**
The student will be able to . . .
- order numerically 3 unit fractions from least to greatest or greatest to least (A)
- solve word problems related to skills learned in this module (A)

4.06.05 **Improper fractions and mixed numbers**
The student will be able to . . .
- identify and define improper fractions and mixed numbers (A)
- change an improper fraction to a mixed number or a whole number (B)
- change a mixed number or a whole number to an improper fraction (B)

4.06.06 **Addition and subtraction of mixed numbers with common denominators**
The student will be able to . . .
- add mixed numbers with common denominators (B)
- subtract mixed numbers with common denominators (B)
- solve word problems related to skills learned in this module (B)

4.06.07 **Common factors**
The student will be able to . . .
- find the common factors of two or more whole numbers less than 100 (B)
- find the greatest common factor of a set of whole numbers (B)
- find the common multiples of two or more whole numbers less than 100 (B)
- find the least common multiple of a set of whole numbers (B)

4.06.08 **Addition and subtraction of fractions with unlike denominators**
The student will be able to . . .
- find equivalent fractions with a common denominator (B)
- add fractions having unlike denominators (B)
- subtract fractions having unlike denominators (B)
- solve word problems related to skills learned in this module (B)

4.06.09 **Ratio**
The student will be able to . . .
- write a ratio to represent a relationship between two quantities (A)
- solve word problems related to skills learned in this module (A)

5.06.01 **Equivalent fractions**
The student will be able to . . .
- use the equal multiplication property to generate equivalent fractions (A)
- rename fractions as simpler equivalent fractions in lower terms by dividing the numerator and the denominator by the same number (A)
- express fractions in simplest form (lowest terms) (B)
- arrange a set of fractions from the least to the greatest (B)

5.06.02 **Improper fractions and mixed numerals**
The student will be able to . . .
- change an improper fraction to a mixed numeral of a whole number (A)
- change a mixed numeral or a whole number to an improper fraction (A)

5.06.03 **Multiplication of fractions**

The student will be able to . . .

- multiply a whole number by a number expressed as a fraction and rename it in lowest (simplest) terms (A)
- find a fractional part of a whole number by creating equal parts (given a fraction whose denominator is a divisor of the whole number) (A)
- explain that the product of any unit fraction (fraction having a numerator of one) and a whole number can be interpreted as finding that fractional part of the number; and find such products when the whole number is a multiple of the denominator of the fraction (A)
- multiply a unit fraction by a second fraction by extending the understanding of taking a fractional part (B)
- multiply **any** two fractions less than one by extending the understanding of taking a fractional part (B)
- make a picture or other graphic illustration to demonstrate multiplication of fractions less than one (A)
- multiply with fractions having values less than one by taking products of numerators and denominators (B)
- create and solve word problems related to skills learned in this module (B)

5.06.04 **Greatest common factor**

The student will be able to . . .

- find the greatest common factor (GCF) of two or more numbers (A)
- rename fractions in **lowest** terms using the GCF (A)
- express computational results in lowest terms (A)

5.06.05 **Addition and subtraction of mixed numbers with common denominators**

The student will be able to . . .

- add mixed numbers with common denominators and express the sum in lowest terms (A)
- subtract mixed numbers with common denominators and express the difference in lowest terms (A)
- subtract mixed numbers with common denominators with regrouping (exchange) and express the difference in lowest terms (A)
- solve word problems related to skills learned in this module (A)

5.06.06 **Least common multiple**

The student will be able to . . .

- find the least common multiple (LCM) of two or more numbers (A)
- rename two or more fractions as fractions with common denominators (A)

5.06.07 **Addition and subtraction of fractions with unlike denominators**

The student will be able to . . .

- add fractions with unlike denominators and express the sum in lowest terms (A)
- subtract fractions with unlike denominators and express the difference in lowest terms (A)
- solve word problems related to skills learned in this module (A)

5.06.08 **Addition and subtraction of mixed numerals with unlike denominators**
The student will be able to . . .
- add mixed numerals with unlike denominators and express the sum in lowest terms (B)
- subtract mixed numerals with unlike denominators and express the difference in lowest terms (B)
- solve problems related to skills learned in this module (B)

5.06.09 **Multiplication with mixed numerals**
The student will be able to . . .
- multiply a whole number by a mixed numeral and express the product in simplest terms (A)
- multiply two mixed numerals and express the product in simplest terms (B)
- solve word problems related to skills learned in this module (A)

5.06.10 **Division of a whole number by a fraction**
The student will be able to . . .
- divide a whole number by a fraction using representative materials (A)
- divide a whole number by a fraction by renaming the whole number as a fraction with the same denominator (B)
- create and solve word problems related to skills learned in this module (B)

6.06.01 **Factor primes**
The student will be able to . . .
- rename even numbers greater than 2 as the sum of two primes (B)
- rename composite numbers as the product of primes (A)
- use prime factorization to find the greatest common factor (A)

6.06.02 **Estimation with fractions**
The student will be able to . . .
- estimate the result of addition and subtraction of fractions before computation (A)
- solve word problems related to skills learned in this module (A)

6.06.03 **Subtraction of fractions**
The student will be able to . . .
- subtract a mixed number from a whole number with simplification (A)
- subtract fractions having unlike denominators with simplification (A)
- check solutions using the principle of related operations (B)
- solve word problems related to skills learned in this module (A)

6.06.04 **Ratio**
The student will be able to . . .
- compare two numbers by naming the ratio of one to another and write them as a fraction (A)
- write equivalent ratios for a given ratio (A)

6.06.05 **Properties of multiplication with fractions**
The student will be able to . . .
- recognize that the commutative property of multiplication can be applied to multiplication of fractions (A)
- recognize that the distributive property of multiplication can be applied to multiplication of fractional numbers (A)
- estimate and compute the result of multiplication of fractions and mixed numbers (B)
- solve word problems related to skills learned in this module (A)

6.06.06 **Division of fractions**
The student will be able to . . .
- estimate the result of division of fractions before computation (A)
- divide a fraction by a fraction (A)

6.06.07 **Division of mixed numbers**
The student will be able to . . .
- divide a mixed number by a fraction (A)
- divide a mixed number by a mixed number (A)
- solve word problems related to skills learned in this module (A)

7.06.01 **Advanced fraction computations and concepts**
The student will be able to . . .
- find the average of two or more fractions (C)
- raise a fraction to an indicated power (C)

7.06.02 **Division with fractions**
The student will be able to . . .
- compute the quotient of two fractions using the algorithm:
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$$ (A)
- solve verbal problems related to skills learned in this module (A)

7.06.03 **Ratio and proportion**
The student will be able to . . .
- compare two numbers by naming the ratio of one to another (A)
- write equivalent ratios for a given ratio (A)
- identify a proportion as a statement of equality between two ratios (A)
- identify means and extremes in a proportion (A)
- determine if two ratios form a true proportion (A)
- solve verbal problems related to skills learned in this module (A)

7.06.04 **Meaning of percent**
The student will be able to . . .
- convert a fraction whose denominator is a factor of 100 to a decimal and a percent (A)
- convert a mixed number to a decimal and a percent (A)
- convert a whole number percent to decimal fraction (A)

7.06.05 **Verbal problems involving percent**
The student will be able to . . .
- solve a verbal problem to find the amount of discount (A)
- solve a verbal problem to find the amount of sales tax (A)
- solve a verbal problem to find simple interest (Interest = Principal × Rate × Time) (A)

8.06.01 **Properties of fractions**
The student will be able to . . .
- use the commutative and associative properties of addition with fractions (A)
- use the commutative and associative properties of multiplication with fractions (A)
- recognize the additive identity property and the additive identity element 0 (A)
- recognize the multiplicative identity property and the multiplicative identity element 1 (A)
- recognize the additive inverse property and the additive inverse of a fraction (A)
- recognize the multiplicative inverse property and the multiplicative inverse element of a fraction (A)
- use the distributive property of multiplication over addition with fractions (A)

8.06.02 **Positive and negative fractions**

The student will be able to . . .

- find negative and positive fractions on a number line (A)
- determine that $-\frac{a}{b}$ is equivalent to $-a/b$ and $a/-b$ (A)
- compare two fractions and/or two mixed numbers using $>$, $<$ and $=$ (B)
- order a set of fractions (B)

8.06.03 **Proportions**

The student will be able to . . .

- solve a verbal problem with proportions and price rates (A)
- solve a verbal problem involving proportion and rate, time and distance (A)
- solve verbal problems involving proportion and scale drawings (A)
- solve a verbal problem involving proportions (A)

8.06.04 **Percents**

The student will be able to . . .

- convert a mixed number or decimal to a percent greater than 100% (A)
- convert a percent greater than 100% to a mixed number or decimal (A)
- convert a decimal to a percent less than 1% (A)
- convert a percent less than 1% to a decimal (A)
- recognize common fraction/decimal equivalents (A)

8.06.05 **Using percents to solve problems**

The student will be able to . . .

- use percents to solve verbal problems involving compound interest (A)
- use percents to solve a problem found in a newspaper (A)

DECIMALS

Topic 07

3.07.01 **Decimals**
The student will be able to . . .
- identify and read a decimal (B)
- write tenths as decimals (B)
- count by tenths (B)

3.07.02 **Addition and subtraction of decimals to tenths**
The student will be able to . . .
- add decimals in horizontal and vertical formats without regrouping (exchange) (B)
- subtract decimals in horizontal and vertical formats without regrouping (exchange) (B)
- solve word problems related to skills learned in this module (B)

4.07.01 **Decimals to hundredths**
The student will be able to . . .
- identify and read a decimal (A)
- write tenths as decimals (A)
- count by tenths (A)
- write any fraction with a denominator of 100 as a decimal (A)
- identify the place value of digits in decimals through hundredths (A)

4.07.02 **Order decimals**
The student will be able to . . .
- compare and order two or more decimals using the relation symbols < (less than), > (greater than), = (equal to) (A)
- recognize equivalency of decimals in tenths and hundredths (A)

4.07.03 **Addition and subtraction of decimals to hundredths**
The student will be able to . . .
- add decimals to hundredths (A)
- subtract decimals to hundredths (A)
- solve word problems related to skills learned in this module (A)

4.07.04 **Multiplication and division of decimals to tenths**
The student will be able to . . .
- multiply decimals to tenths with products to hundredths (A)
- divide decimals with dividends to tenths by whole number divisors (A)
- solve word problems related to skills learned in this module (A)

5.07.01 **Decimals—thousandths**
The student will be able to . . .
- identify the place value of decimals through thousandths (A)
- write tenths, hundredths, and thousandths as decimals (A)
- compare decimals expressed as tenths, hundredths, and thousandths, using the symbols >, <, = (B)

5.07.02 **Rounding decimals**
The student will be able to . . .
- round decimals to the nearest whole number (A)
- round decimals to the nearest tenth (B)
- round decimals to the nearest hundredth (B)

5.07.03 **Addition and subtraction with decimals through thousandths**
The student will be able to . . .
- estimate sums and differences by rounding the nearest whole number (A)
- add and subtract with decimals through thousandths and check (A)
- solve word problems related to skills learned in this module (A)

5.07.04 **Multiplication with decimals**
The student will be able to . . .
- multiply decimals by 10 and 100 (A)
- multiply decimals (yielding products no smaller than one thousandth) (A)
- solve word problems related to skills learned in this module (A)

5.07.05 **Division with decimals**
The student will be able to . . .
- divide a dividend expressed as a decimal by a whole number divisor and check (A)
- divide decimals by 10 and 100 (A)
- use the multiplicative identity (1) to express a quotient of decimals as an equivalent division by a whole number (B)
- divide a dividend expressed as a decimal by a divisor expressed as a decimal by solving an equivalent problem involving division by a whole number (B)
- solve problems related to skills learned in this module (A)

6.07.01 **Decimal notation**
The student will be able to . . .
- change fractions with denominators that are factors of multiples of ten to equivalent decimals (1/2, 1/5, 1/4, 1/20, 1/25, 1/50, etc.) (A)
- write any proper fraction or mixed numeral with a denominator of 1000 as a decimal fraction (A)
- change fractions with denominators unrelated to multiples of ten, to equivalent decimals (1/3, 1/8, 1/6, etc.) (A)

6.07.02 **Compare numbers in decimal notation**
The student will be able to . . .
- compare a given decimal by hundredths or thousandths (A)
- solve word problems related to skills learned in this module (A)

6.07.03 **Round decimals**
The student will be able to . . .
- round decimals in hundredths to the nearest tenth (A)
- round decimals in thousandths to the nearest hundredth (A)

6.07.04 **Percent**
The student will be able to . . .
- write a percent as a decimal (A)
- express fractions and decimals as an equivalent percent (A)
- solve word problems related to skills learned in this module (A)

6.07.05 **Add numbers in decimal form**
The student will be able to . . .
- add decimals through ten thousandths and check (A)
- solve word problems related to skills learned in this module (A)

6.07.06 **Subtract numbers in decimal form**
The student will be able to . . .
- subtract decimals through ten thousandths and check (A)
- solve word problems related to skills learned in this module (A)

6.07.07 **Multiplication of decimals**
The student will be able to . . .
- multiply a decimal by 10, 100 or 1000 (A)
- multiply a whole number by a decimal (A)
- multiply a decimal by a decimal (A)
- solve word problems related to skills learned in this module (A)

6.07.08 **Division of decimals**
The student will be able to . . .
- divide decimals by a whole number (A)
- divide decimals by a decimal fraction (A)
- solve word problems related to skills learned in this module (A)

7.07.01 **Decimal relationships**

The student will be able to . . .

- extend the base ten place value chart to include decimals whose denominators are powers of 10, and read and write decimal fractions in numerical and word form (A)
- identify equivalent decimals (A)
- compare and order decimals (A)
- round off a given decimal to a specified place (A)
- solve word problems related to skills learned in this module (A)

7.07.02 **Addition of decimals**

The student will be able to . . .

- compute the sum of two or more decimals arranged horizontally (A)
- solve word problems related to skills learned in this module (A)

7.07.03 **Subtraction of decimals**

The student will be able to . . .

- compute the difference of two decimals arranged vertically with the minuend having fewer decimal places than the subtrahend (A)
- solve word problems related to skills learned in this module (A)

7.07.04 **Multiplication of decimals**

The student will be able to . . .

- determine and place the decimal point in the product of a given multiplication example if the digits of the product are given (A)
- solve word problems related to skills learned in this module (A)

7.07.05 **Division of decimals**

The student will be able to . . .

- compute the quotient by correctly placing the decimal point given a division example in which the divisor is a power of 10 (A)
- compute the quotient to a specified decimal place given a division example in which the divisor is a whole number or a decimal (A)
- solve word problems related to skills learned in this module (A)

7.07.06 **Convert to decimals or fractions**

The student will be able to . . .

- convert a fraction to a terminating or non-terminating (repeating) decimal fraction (A)
- convert a terminating or non-terminating (repeating) decimal to a fraction (A)

8.07.01 **Scientific notation**

The student will be able to . . .

- write a standard base ten numeral in scientific notation (B)
- write a number expressed in scientific notation as a standard base ten numeral (B)
- multiply two numbers expressed in scientific notation (B)
- divide two numbers expressed in scientific notation (B)

8.07.02 **Rational numbers/terminating decimals**

The student will be able to . . .

- redefine a terminating decimal as a repeating decimal with a "0 repetend" (A)
- define every rational number as a repeating decimal (A)

8.07.03 **Irrational numbers**

The student will be able to . . .

- define an irrational number as a non-repeating, non-terminating decimal (A)
- distinguish between a rational and irrational number (A)
- approximate to the nearest tenth the square root of non-perfect squares (A)
- solve verbal/numerical problems related to skills learned in this module (A)

GRAPHS, PROBABILITY, STATISTICS

Topic 12

0.12.01 **Number relationships**
The student will be able to . . .
- construct a bar graph using blocks/cubes
- transfer a graph to a pictorial representation
- interpret pictorial representations as those representing concrete objects or people
- identify number relationships using symbols and pictures
- count aloud the number of squares that are colored on a bar graph
- color the corresponding number of squares on a vertical bar graph when given concrete objects or the pictorial representation
- write the numeral that represents the number of squares colored on a vertical and horizontal bar graph

0.12.02 **Graph reading**
The student will be able to . . .
- construct and interpret a simple picture graph

0.12.03 **Chance events/predicting**
The student will be able to . . .
- distinguish between certain and uncertain in regard to events
- distinguish between more or less likely in chance situations
- recognize the outcome of activities as chance (guessing)

1.12.01 **Pictographs**
The student will be able to . . .
- collect, organize and graph data relating to preference using two categories at a time (A)
- read and interpret the information given in a graph using more than two categories (A)
- use information on a graph to answer questions involving more or less (A)

1.12.02 **Bar graphs**
The student will be able to . . .
- color the corresponding number of squares on vertical or horizontal bar graphs, when given concrete objects or the pictorial representation (B)
- identify and record the number of squares that are colored on a bar graph (B)
- gather, organize and record data based on personal or group experiences to complete a bar graph using one-to-one correspondence (B)

1.12.03 **Probability—chance events**
The student will be able to . . .
- predict the likelihood of events happening (B)
- record the outcome of a chance event (B)

2.12.01 **Data—collect, tabulate, arrange and compare in tables**
The student will be able to . . .
- collect, tabulate and arrange data in a table (A)
- compare data in a table (A)

2.12.02 **Bar graphs**
The student will be able to . . .
- collect and structure data based on personal or group experiences to complete a bar graph (A)
- interpret a bar graph (A)

2.12.03 **Probability—predict and tally outcomes**
The student will be able to . . .
- predict the outcome of three equally likely events (B)
- tally the outcome in a three-possibility event (B)

2.12.04 **Arrangements of 2 through 5 items**
The student will be able to . . .
- make arrangements of 2 through 5 items using all the items (B)
- define and identify subsets of a given set of objects (B)
- make subsets of 1 or 2 objects from sets of 3 or 4 concrete objects (B)

2.12.05 **Ratio—2:1 correspondence**
The student will be able to . . .
- use two-to-one correspondence to develop the concept of ratio (B)
- use a chart to show ratio (B)

3.12.01 **Bar graphs**
The student will be able to . . .
- read and interpret a bar graph (A)
- construct a bar graph (A)

3.12.02 **Probability**
The student will be able to . . .
- predict the outcomes of equally likely events, and compare the results with the predictions (A)
- predict the outcome of unequally likely events (A)

3.12.03 **Arrangements—tree diagrams**
The student will be able to . . .
- list and count the number of possible arrangements in a set of objects (A)
- make a tree diagram to display combinations of given data (A)

4.12.01 **Graphs**
The student will be able to . . .
- make frequency tables from tallied data (A)
- graph data drawn from newspapers, magazines, polls, and appropriate activities (A)
- read and interpret information from a line graph (B)
- construct a line graph from a table of data (B)

4.12.02 **Statistics**
The student will be able to . . .
- find the mean (average) from given data (A)
- solve word problems involving finding the averages of given numbers (A)
- define the range as the difference between the largest and the smallest numbers in a set (A)
- identify the mode as the number that occurs most frequently (B)

4.12.03 **Probability**
The student will be able to . . .
- determine probabilities by comparing favorable outcomes with total possible outcomes and express as fractions (A)
- determine probabilities where the outcomes are not equally likely and express as fractions (A)
- predict the outcomes of probability experiments (A)

4.12.04 **Arrangements**
The student will be able to . . .
- identify all the possible arrangements for a group of objects (A)
- solve word problems related to skills learned in this module (A)

5.12.01 **Statistics—mean and median**
The student will be able to . . .
- find the **range** and **mean** when given a set of numerical data (A)
- find the **median** of the set when given an **odd** number of values (B)

5.12.02 **Line graphs**
The student will be able to . . .
- read and interpret information from a line graph (A)
- construct a line graph from a table of data (A)

5.12.03 **Circle graphs**

The student will be able to . . .

- read and interpret a circle graph (A)
- use a protractor to construct circle graphs where the sectors represent fractions with like denominators (B)
- use a protractor to construct circle graphs where the sectors represent fractions with unlike denominators (B)

5.12.04 **Comparing graphs**

The student will be able to . . .

- compare data presented in various graphic formats (A)
- construct various types of graphs to illustrate the same set of data (C)

5.12.05 **Probability concepts**

The student will be able to . . .

- for a given experiment identify events which are impossible, certain, or which sometimes occur (A)
- identify, among events that may occur for a given experiment, those which are **equally likely** to occur and those which have **different chances** of occurring (A)
- recognize a **sample space** for a given experiment (A)
- for an experiment which has a sample space of equally likely outcomes, express the probability of an event as the fraction:

$$\frac{\text{number of favorable outcomes possible}}{\text{number of outcomes in the sample space}}$$ (A)

- conduct probability experiments where each outcome is equally likely to occur, and compare the results with predictions (A)

5.12.06 **Simple probability relationships**

The student will be able to . . .

- design and conduct probability experiments where the outcomes are not equally likely to occur, and compare results to the predictions (A)
- state that, for a given experiment, the sum of the probabilities of all possible outcomes will always be one (A)
- describe the relationship between the probability of event "A" and the probability of "not A"; use this relationship to calculate probability (A)

5.12.07 **Double bar graphs**

The student will be able to . . .

- read and interpret a double bar graph (A)
- construct a double bar graph from a set of data (B)

6.12.01 **Graphs**

The student will be able to . . .

- interpret and construct circle graphs from a table of data (A)
- construct and compare bar, line and circle graphs and note the advantages and disadvantages of each (A)

6.12.02 **Statistics**

The student will be able to . . .

- find the mode of a given set of data (A)
- find the median of a given set of data (A)
- identify the mean, median and mode after involvement in an extended recording project (A)

6.12.03 **Compound probability**
The student will be able to . . .
- determine the probability of one outcome from a sample space (A)
- determine the probability of two independent events occurring together from a sample space (A)

6.12.04 **Coordinate systems**
The student will be able to . . .
- locate, read, and interpret coordinates on a graph (C)
- graph points whose coordinates include positive and negative numbers using all 4 quadrants (C)

6.12.05 **Permutations and combinations**
The student will be able to . . .
- count the number of permutations that can be made with a set of objects (A)
- use the multiplication principle to determine the number of permutations that can be made with a set of objects (A)
- count the number of groups when making combinations (A)

7.12.01 **Probability**
The student will be able to . . .
- construct a sample space for various experiments (A)
- read and use the expression P(A) to represent the probability of event A (A)
- write the probabilities involved with simple experiments: flipping a coin, rolling a die (A)

7.12.02 **Probability with a spinner**
The student will be able to . . .
- state that the sum of the probabilities in a given experiment is always 1 (A)
- state and use the meaning of equally likely events (A)
- state and use the meaning of complementary events to their probabilities (A)

7.12.03 **Multiple graphs**
The student will be able to . . .
- construct, read and interpret a multiple bar graph (involving a two bar comparison) (B)
- read and interpret a multiple line graph (involving a two line comparison) (B)
- construct, read and interpret a circle graph (involving fractional or percent comparisons) (B)

7.12.04 **Frequency tables**
The student will be able to . . .
- obtain the mode from the frequency table (A)
- recognize how changes in data affect the mean, median and mode (A)

7.12.05 **Compound probability**
The student will be able to . . .
- determine the probability of two independent events occurring together by using a sample space (B)
- determine the probability of either of two outcomes occurring by using a sample space (B)

8.12.01 **Multiple graphs**

The student will be able to . . .

- construct, read and interpret a multiple bar graph (involving a two bar comparison) (A)
- construct, read and interpret a multiple line graph (involving a two line comparison) (A)
- construct, read and interpret a circle graph (involving fractional or percent comparisons) (A)

8.12.02 **Statistics**

The student will be able to . . .

- read, interpret and make a histogram (A)
- make a frequency polygon from a histogram (A)

8.12.03 **Comparison of permutations and combinations**

The student will be able to . . .

- determine a permutation of n things taken r at a time where $r \leq n$ (A)
- compare the number of arrangements in a permutation with the number of arrangements in a combination (A)

8.12.04 **Compound probability**

The student will be able to . . .

- determine the probability of two independent events occurring together by using a sample space and a formula (A)
- determine the probability of either of two outcomes occurring by using a sample space and a formula (A)
- determine the probability of more than two independent outcomes occurring at the same time using a formula (B)
- determine the probability of exactly, at least or at most r successes in n trials by experimentation (B)

PROBLEM SOLVING

Topic 13

0.13.01 **Problem solving: language**
The student will be able to . . .
- locate and describe objects in the classroom
- locate and describe objects within a picture
- make and explain inferences based upon information obtained from a picture

0.13.02 **Problem solving: one-to-one correspondence**
The student will be able to . . .
- match pictures and objects
- solve simple problems using one-to-one correspondence

1.13.01 **Use drawings to represent objects**
The student will be able to . . .
- fill in the missing part of a picture (A)
- make drawings to represent objects in a picture (A)

1.13.02 **Develop mathematical language through questions**
The student will be able to . . .
- identify numerical information within a picture (A)
- distinguish between a fact and a question (A)
- develop questions based upon information obtained from a picture (A)

1.13.03 **Use tables to develop questioning skills**
The student will be able to . . .
- collect and record data on a table (A)
- develop questions based upon information obtained from a table (A)

1.13.04 **Solve story problems**
The student will be able to . . .
- solve story problems through dramatization (A)
- solve story problems using a number line (B)

1.13.05 **Story problems/number sentences**
The student will be able to . . .
- translate a story problem into a number sentence (A)
- create a story problem given a number fact through 9 (A)

1.13.06 **Problem solving: patterns**
The student will be able to . . .
- identify the rule for a non-numerical pattern, given the pattern (A)
- identify the rule for a numerical pattern, given the pattern (A)
- identify patterns for sums and differences using concrete materials and tables (B)

2.13.01 **Locate information**
The student will be able to . . .
- locate information in the environment in order to solve simple problems (A)
- locate information in a picture in order to solve simple problems (A)
- locate information on a table or chart in order to solve simple problems (A)

2.13.02 **Guess and check**
The student will be able to . . .
- check guesses against problem conditions in non-numerical problems (A)
- check guesses against problem conditions in numerical problems (A)

2.13.03 **Use data**
The student will be able to . . .
- answer questions based upon a set of data (A)
- create questions based upon a set of data (A)
- make inferences based upon a set of data (A)

2.13.04 **Patterns**
The student will be able to . . .
- create and explain a pattern using concrete material (A)
- complete a picture pattern and state the rule (A)
- complete a numerical pattern and state the rule (A)

2.13.05 **Missing information**
The student will be able to . . .
- determine whether sufficient information exists to solve a problem (A)
- supply missing information to solve a problem (A)

3.13.01 **Guess and check**
The student will be able to . . .
- make and verify a guess (A)
- check his/her guess against problem conditions (A)

3.13.02 **Locate information**
The student will be able to . . .
- locate pertinent information within a chart/table to solve problems (A)
- locate pertinent information within a paragraph to solve problems (A)

3.13.03 **Patterns**
The student will be able to . . .
- complete a numerical pattern and state the rule for the pattern (A)
- arrange numbers to form a pattern and state the rule for the pattern (A)

3.13.04 **Create models/charts to solve problems**
The student will be able to . . .
- use representative materials to help solve a problem (A)
- create charts/tables to help solve problems (A)
- develop questions based upon information obtained from a model/chart (A)

4.13.01 **Problem solving: use tables and graphs to locate information**
The student will be able to . . .
- locate information on a table/chart and solve problems using this information (A)
- locate information on a graph and solve problems using this information (A)

4.13.02 **Problem solving: use number sentences to solve problems**
The student will be able to . . .
- translate a one step word problem into an open number sentence and solve (A)
- choose and sequence operations to solve a two step word problem (A)
- create a word problem based upon a solution (B)

4.13.03 **Problem solving: guess and check**
The student will be able to . . .
- check guesses against problem conditions involving place value problems (A)
- check guesses against problem conditions involving addition and subtraction problems (A)
- check guesses against problem conditions involving multiplication and division problems (A)

4.13.04 **Problem solving: patterns**
The student will be able to . . .
- locate patterns on a calendar (A)
- locate patterns on a chart/table (A)

5.13.01 **Problem solving: draw a picture/make a table**
The student will be able to . . .
- draw a picture to help solve a problem (A)
- make a table to solve a problem (A)
- create and solve problems for which pictures or charts are helpful tools (B)

5.13.02 **Problem solving: using prior knowledge**
The student will be able to . . .
- identify the prior knowledge needed to solve a problem and use that knowledge to solve the problem (A)
- create and solve problems where prior knowledge is needed (B)

5.13.03 **Problem solving: solve a simpler problem**
The student will be able to . . .
- solve problems by creating and solving similar, but simpler, problems (A)
- create and solve a problem which can be solved by using a simpler problem (A)

5.13.04 **Problem solving: working backward**
The student will be able to . . .
- solve problems by working backward (A)
- create a problem which can be solved by working backward (C)

6.13.01 **Organize data to perceive patterns**
The student will be able to . . .
- recognize patterns to solve problems (A)
- create a simple chart in order to recognize a pattern (A)
- organize data on a chart to perceive patterns to solve problems (A)

6.13.02 **Use number sentences to solve problems with two or more steps**
The student will be able to . . .
- choose and sequence operations to solve a two step word problem (A)
- choose and sequence operations to solve a three step world problem (A)

6.13.03 **Organize data into tables and charts**
The student will be able to . . .
- complete a table of missing information (A)
- organize data into a table in order to solve a problem (A)

6.13.04 **Use models to solve problems**
The student will be able to . . .
- use a model drawing to solve a problem (A)
- create questions given a model (A)

7.13.01 **Solving a simpler but equivalent problem**
The student will be able to . . .
- use models/chart to simplify a problem (A)
- solve problems by simplifying the data by finding a pattern (A)

7.13.02 **Working backwards**
The student will be able to . . .
- make a model/chart of data in a problem (A)
- use a model/chart to solve a problem by working backwards (A)

7.13.03 **Guess and check**
The student will be able to . . .
- make and refine a guess about his/her environment (A)
- check guesses about problem conditions (A)

7.13.04 **Patterns**

The student will be able to . . .

- identify and extend a numerical pattern (A)
- use patterns to solve problems (A)

8.13.01 **Selection by elimination**

The student will be able to . . .

- use information given in a problem to eliminate all possible solutions except the correct one (A)
- list all possible solutions and then eliminate all but the correct one (A)

8.13.02 **Models**

The student will be able to . . .

- make a picture-type model to solve a problem (A)
- select a physical or paper-pencil type model to solve a problem (A)

8.13.03 **Tables**

The student will be able to . . .

- make a list to solve a problem (A)
- find a rule from a table (A)
- use a table to generate information not given to solve a problem (A)

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